THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS.*

PART II.

§ III.

The following notes, on certain MSS. which Gerhardt does not give in full, are taken from G. 1848, p. 20 et seq. (see also G. 1855, p. 55 et seq.)

In a manuscript of August, 1673, bearing the title *Methodus nova investigandi Tangentes linearum curvarum ex datis applicatis, vel contra Applicatis ex datis productis, reductis, tangentibus, perpendicularibus, secantibus,* Leibniz begins at once with an attempt to find a method that is applicable to any curve for the determination of its tangent. "But if," says Leibniz with regard to the classification of curves which Descartes laid down as fundamental for his method of tangents, "the figure is not geometrical — such as the cycloid—it does not matter; for it will be treated as an example of a geometrical curve, by supposing that there is a relation between the straight lines and curves by which they are made known to us; in this way, tangents can be drawn just as well to either geometrical or ageometrical curves, as far as the nature of the figure allows." He considers the curve as a polygon with an infinite number of sides, and here already he constructs what he calls the "Characteristic Triangle," whose sides are an infinitely small arc of the curve, and the differences between the ordinates and between the abscissae; this is similar to the triangle whose sides are the tangent, the subtangent and the ordinate for the point of contact. In just the same manner as used by Descartes, Leibniz seeks the tangent by means

* Part I appeared in *The Monist* of October, 1916.

of the subtangent; he denotes the infinitely small differences of the abscissae by b, and verifies for the parabola, that his method works out correctly, when the terms of the equation that contain the infinitely small quantities are neglected. The omission of these terms, however, does not appear to Leibniz to be a method to be relied upon. In fact, he says: "It is not safe to reject multiples of the infinitely small part b, and other things; for it may happen that through the compensation of these with others,[1] the equation may come to a totally different condition." So he seeks to obtain the determination of the subtangent in some other way. "The whole question is, how the applied lines can be found from the differences of two applied lines," are his own words. He then finds that the solution of this problem reduces to the summation of a series, of which the terms are the differences of consecutive abscissae.

At the end of the manuscript Leibniz proceeds to speak of the inverse problem: "It is an important subject for investigation, whether it is possible, by retracing our steps, to proceed from tangents and other functions to ordinates. The matter will be most accurately investigated by tables[2] of equations; in this way we may find out in how many ways some one equation may be produced from others, and from that which of them should be chosen in any case. This is, as it were, an analysis of the analysis itself, but if that is done it forms the fundamental of human science, as far as this kind of things is concerned." Ultimately Leibniz obtains the following result: "The two questions, the first that of finding the description of the curve from its elements, the second that of finding the figure from the given differences, both reduce to the same thing. From this fact it can be taken that almost the whole of the theory of the inverse method of tangents is reducible to quadratures."

According to this, Leibniz has in the middle of the year 1673 already attained to the knowledge that the direct and the so-called inverse tangent-problem have an undoubted connection with one another; he has an idea that the latter may be capable of reduction to a quadrature (i. e., to a summation).

Again, in a manuscript dated October 1674, i. e., fourteen months later, which bears the title *Schediasma de Methodo Tan-*

[1] It is impossible to see, without a fuller knowledge of the context, whether this refers to "compensation of errors," or whether Leibniz is alluding to the possibility of all the finite terms cancelling one another.

[2] Leibniz comes back to this point later; see § IV.

gentium inversa ad circulum applicata, he is able to say for certain that "the quadratures of all figures follow from the inverse method of tangents, and thus the whole science of sums and quadratures can be reduced to analysis, a thing that nobody even had any hopes of before."

After Leibniz thus recognized the identity between the inverse tangent-problem, of which the general solution had not been found by Descartes, and the quadrature of curves, he applied himself to the investigation of series by the summation of which quadratures were then obtained. In a very extensive discussion, bearing the date of October, 1674, and the title *Schediasma de serierum summis, et seriebus quadraticibus,* Leibniz starts from the series

$$\frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \frac{b^5}{5} - \cdots\cdots,$$

and obtains the following general rule: "By calling the variable ordinates x, and the variable abscissae y, and b the abscissa of the greatest ordinate e, and d the abscissa of the least ordinate h," are Leibniz's own words, "we have the following rules:

$$\frac{x^2}{2} = ywx - \frac{yw^2}{2} + \frac{d^2h}{2},$$

$$\frac{h^2w}{2} + \frac{d^2h}{2} = xy - \frac{x}{2}, \; e-h=w,$$

$$xw = \frac{e^2}{2} - \frac{w^2}{2},$$

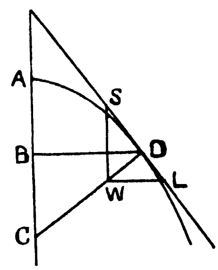
$yw = x$ in decreasing values, for in ascending or increasing values $yw = eb - x$."[3]

Leibniz then goes on to remark: "These rules are to be altered slightly according as the series increase or decrease; also mention of the least ordinate may be omitted, if it is always understood to be the last ordinate; on the other hand, w can always be inserted wherever mention is made of w. All series hitherto found are contained in the one by means of these rules, except the series of powers, which is to be obtained by taking differences."

[3] This, without either proof or figure, is a hopeless muddle; and yet it is repeated word for word, without any addition or remark, in Gerhardt's 1855 publication. Goodness knows what the use of it was supposed to be in this form! Unless Leibniz has omitted some length, which he has supposed to be unity, the dimensions are all wrong.

In the same essay, Leibniz makes use of a theorem, which he has probably found to be general at an earlier date, namely:

"Since BC is to BD as WL to SW, therefore BC⌒SW,[4] that is, the sum of every BC [applied to AC], is equal to BD⌒WL, that is, the sum of every BD applied to the base; moreover, the sum of every BD applied to the base is equal to half the square on the greatest BD. Further, it is evident that the sum of every WL is equal to the greatest BD."



Accordingly, Leibniz comes to the further conclusion that the method of Descartes, which uses a subsidiary equation with two equal roots, to solve the general inverse-tangent problem, is unsatisfactory. In a manuscript of January, 1675, Leibniz says: "Thus at last I am free from the unprofitable hope of finding sums of series and quadratures of figures by means of a pair of equal roots, and I have discovered the reason why this argument cannot be used; this has worried me for quite long enough."[5]

§ IV.

The manuscript that comes next in date is one that is given in G. 1855. It really consists of three short notes, (1) a theorem on moments, (2) a continuation of the idea started at the end of the manuscript of August, 1673 (§ III), namely the formation of tables of equations that are derivable from certain standard equations, with the appropriate substitutions for each case, (3) a return to the consideration of moments.

This is the first appearance of the word "moment," but from the context it is evident that Leibniz has done some considerable amount of work upon the idea before. If the theorem that is first given is written in modern notation,

[4] The sign ⌒ signifies multiplication.

[5] Observe that as yet nothing has been said about the area of surfaces of revolution or moments about the axis, although we should expect them to be mentioned in connection with the figure that is given; for the next manuscript shows that in October 1675, Leibniz has already done a considerable amount of work on moments.

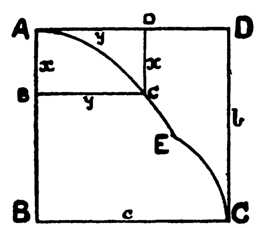
it takes the form of an "integration by parts" and serves to change the independent variable. Thus we have

$$\int xy\, dx = \left[\frac{x^2 y}{2}\right] - \int \frac{x^2}{2}\, dy;$$

and it is readily seen that if x can be expressed as a square root of a simple function of y, as for the circle and the conic sections, then the integral on the right-hand side has no irrationality. This, I take it, is the connection between this theorem and those which follow.

The proof is not so clear as it might be on account of two errors, both I think errors of transcription or misprints. The first *a* should be an x, and the second *a* should be the preposition *a* (= from); also, for modern readers the figure might be improved by showing the *variable* lines AB $(=x)$, BC $(=y)$ as in the accompanying diagram. The argument then is as follows:

Moment of BC$(=y)$ about AD is xy, when it is applied to AB for the summation; for this brings in the infinitesimal breadth of the line.



Moment of DE $(=x)$ about AD is $x^2/2$, when applied to AD, so as to include the infinitesimal breadth of the line, and assuming that the line may be considered to be condensed at its center of gravity. The theorem follows at once.

Note the use of the sign ⊓ as a symbol of equality, which I have allowed to stand in the opening paragraph. Leibniz adopts the ordinary sign two months later, or Ger-

hardt makes the change,[6] so I have not thought it necessary to adhere to it, but only to show it in the opening paragraph.

The only remark that seems to be necessary with regard to the second part of this manuscript is that Weissenborn[7] argues from the continued allusion by Leibniz to the desirability of forming tables of curves whose quadratures may be derived from those of others, especially the conic sections, (starting with the manuscript of November, 1675, where Weissenborn states that it is first hinted), that Leibniz had probably either seen or heard of the *Catalogus curvarum ad conicas sectiones relatarum* of Newton. The point is that Weissenborn seems to have missed the clear reference to the reduction of curves to those of the second degree, in this manuscript of October, 1675. It may of course be just possible that G. 1855, in which this MS. appears, was not at Weissenborn's hand at the time that he wrote, for Weissenborn's book was published in 1856.

With regard to the third part, it will be found in the original Latin that Leibniz, after apparently starting with perfect clearness, gets rather into a muddle toward the end. This is however only apparent, being partly due to an inaccurate figure, and partly to what I am convinced is an error of transcription. This incorrect sentence makes Leibniz write apparently absolute nonsense; but if a correction is made according to the suggestion in the footnote, and reference is made to the corrected diagram that I have added on the right of the figure of Leibniz, as given by Gerhardt, then the proof given by Leibniz reads perfectly smoothly and sensibly.

[6] Gerhardt has a footnote to the effect that, as nearly as possible he has retained the exact form of this and the manuscripts that immediately follow; except in the matter of this one sign I have adhered to the form given by Leibniz.

[7] Weissenborn, *Principien der höheren Analysis*, Halle, 1856.

25 October, 1675.

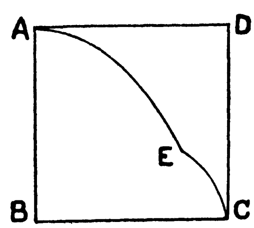
Analysis Tetragonistica Ex Centrobarycis.

Analytical quadrature by means of centers of gravity.

Let any curve AEC be referred to a right angle BAD; let AB $\sqcap$ DC $\sqcap a$,[8] and let the last $x \sqcap b$; also let BC $\sqcap$ AD $\sqcap y$, and the last $y \sqcap c$. Then it is plain that

$$\text{omn. } \overline{yx} \text{ to } x = \frac{b^2c}{2} - \text{omn. } \overline{\frac{x}{2}} \text{ to } y. \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

For, the moment of the space ABCEA about AD is made up of rectangles contained by BC $(= y)$ and AB $(= x)$; also the moment



about AD of the space ADCEA, the complement of the former is made up of the sum of the squares on DC halved $\left(= \frac{x^2}{2}\right)$; and if this moment is taken away from the whole moment of the rectangle ABCD about AD, i. e., from c into omn. x,[9] or from $\frac{b^2c}{2}$, there will remain the moment of the space ABCEA. Hence the equation that I gave is obtained; and, by rearranging it, it follows that

$$\text{omn. } yx \text{ to } x + \text{omn. } \frac{x^2}{2} \text{ to } y = \frac{b^2c}{2} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In this way we obtain the quadrature of the two joined in one in every case; and this is the fundamental theorem in the center of gravity method.

Let the equation expressing the nature of the curve be

$$ay^2 + bx^2 + cxy + dx + ey + f = 0, \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and suppose that $xy = z, \cdots (4)$, then $y = \frac{z}{x}$. $\ldots\ldots\ldots\ldots (5)$

Substituting this value in equation (3), we have

[8] This a should be x.

[9] Here, in the Latin, "*ac* in omn.x" should be "a c in omn.x."

$$\frac{az^2}{x^2}+bx^2+cz+dx+\frac{ez}{x}+f=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

and, on removing the fractions,

$$az^2+bx^4+cx^2z+dx^3+exz+fx^2=0. \quad \ldots\ldots\ldots\ldots \quad (7)$$

Again, let $x^2=2w$ ....(8); then, substituting this value in equation (3), we have

$$ay^2+2bw+cxy+dx+ey+f=0, \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

and therefore

$$x=\frac{-ay^2-2bw-ey-f}{cy+d}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

$$=\sqrt{2w}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

and, squaring each side, we have[10]

$$a^2y^2+4aby^2w+2aey^3+2afy^2+4b^2w^2+4bewy+4bfw$$
$$+e^2y^2+2fey+f^2-2c^2y^2w-4cdyw-2d^2w=0. \quad ..(12)$$

Now, if a curve is described according to equation (7), and also another according to equation (12), I say that the quadrature of the figure of the one will depend on the quadrature of the figure of the other, and *vice versa.*

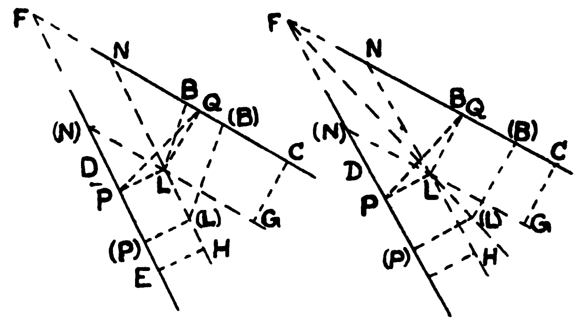
If, however, in place of equation (3), we took another of higher degree, the third say, we should again have two equations in place of (7) and (12); and continuing in this manner, there is no doubt that a certain definite progression of equations (7) and (12) would be obtained, so that without calculation it could be continued to infinity without much trouble. Moreover, from one given equation to any curve, all others can be expressed by a general form, and from these the most convenient can be selected.

If we are given the moment of any figure about any two straight lines, and also the area of the figure, then we have its center of gravity. Also, given the center of gravity of any figure (or line) and its magnitude, then we have its moment about any line whatever. So also, given the magnitude of a figure, and its moments about any two given straight lines, we have its moment about any straight line. Hence also we can get many quadratures from a few given ones. Moreover, the moment of any figure about any straight line can be expressed by a general calculation.

The moment divided by the magnitude gives the distance of the center of gravity from the axis of libration.

[10] In view of this accurate bit of algebra, the faulty work in subsequent manuscripts seems very unaccountable.

Suppose then that there are two straight lines in a plane, given in position, and let them either be parallel or meet, when produced in F. Suppose that the moment about BC is found to be equal to ba^2, and the moment about DE is found to be ca^2. Call the area of the figure v; then the distance of the center of gravity from the straight line BC, namely CG, is equal to $\frac{ba^2}{v}$, and its distance from the straight line DE, namely EH, is equal to $\frac{ca^2}{v}$; therefore CG is to EH as b is to c, or they are in a given ratio.[11]



GERHARDT'S DIAGRAM. SUGGESTED CORRECTION.

Now suppose that the straight line EH, remaining in the plane, traverses the straight line DE, always being perpendicular to it, and that the straight line CG traverses the straight line BC, always perpendicular to it, and that the end G leaves as it were its trace, the straight line G(N), and the end H the straight HN. Then, if BC and DE meet anywhere, G(N) and HN must also meet somewhere, either within or without the angle at F. Let them meet at L; then the angle HLG is equal to the angle EFC, and PLQ (supposing that PL = EH and LQ = CG) will be the supplement of the angle EFC between the two straight lines, and will thus be a given angle. If then PQ is joined, the triangle PQL is obtained, having a given vertical angle, and the ratio of the sides forming the vertex, QL : LP, also given.

When then BL is taken, or (B)(L), of any length whatever, since the angle BLP always remains the same, and in addition we have BL to LP as (B)(L) to (L)(P), therefore also BL to (B)(L) as LP to (L)(P); and this plainly happens when FL is also propor-

[11] This proves the fundamental theorem given lower down, with regard to a pair of parallel straight lines; and he now goes on to discuss the case of non-parallel straight lines.

tional to these, that is, when a straight line passes through F, L, (L),......

Hence, since we are not here given several regions, it follows that the locus is a straight line. Therefore, given the two moments of a figure about two straight lines that are not parallel,........., the area of the figure will be given, and also its center of gravity.[12]

Behold then the fundamental theorem on centers of gravity. If two moments of the same figure about two parallel straight lines are given, then the area of the figure is given, but not its center of gravity.

Since it is the aim of the center of gravity method to find dimensions from given moments, we have hence two general theorems:

If we are given two moments of the same figure about two straight lines, or axes of libration, that are parallel to one another, then its magnitude is given; also when the moments about three non-parallel straight lines are given. From this it is seen that a method for finding elliptic and hyperbolic curves from given quadratures of the circle and the hyperbola is evident.[13] But of this in a special note.

§ V.

The next manuscript to be considered is a continuation of the preceding, and is dated the next day. Its character is of the nature of disjointed notes, set down for further consideration.

[12] The passage in Gerhardt reads:

Datis ergo duobus momentis figurae ex duabus rectis non parallelis, dabitur figurae momentis tribus axibus librationis, qui non sint omnes paralleli inter se, dabitur figurae area, et centrum gravitatis.

For this I suggest:

Datis ergo *tribus* momentis figurae ex *tribus* rectis non parallelis, *aliter* figurae momentis tribus axibus librationis, qui non *sunt* omnes paralleli inter se....

The passage would then read:

Given three moments of a figure about three straight lines that are not parallel, in other words, the moments of the figure about three axes of libration, which are not all parallel to one another, then the area of the figure will be given and also the center of gravity.

If the alternative words are *written* down, one under the other, and not too carefully, I think the suggested corrections will appear to be reasonable.

[13] Apparently, here Leibniz is referring back to the theorem at the beginning of the article.

26 October, 1675.

Another tetragonistic analysis can be obtained by the aid of curves. Thus, let the same curve be resolved into different elements, according as the ordinates are referred to different straight lines. Hence also arise diverse plane figures, consisting of elements similar to the given curve; and since all of these are to be found from the given dimension of the curve, it follows that from the dimension of any one of the curves of this kind the rest are obtained.

In other ways it is possible to obtain curves that depend on others, if to the given curve are added the ordinates of figures of which the quadrature is either known or can be obtained from the quadrature of the given one.

Just as areas are more easily dealt with than curves, because they can be cut up and resolved in more ways, so solids are more manageable than planes and surfaces in general. Therefore, whenever we divert the method for investigating surfaces to the consideration of solids, we discover many new properties; and often we may give demonstrations for surfaces by means of solids when they are with difficulty obtained from the surfaces themselves. Tschirnhaus observed in a delightful manner that most of the proofs given by Archimedes, such as the quadrature of the parabola, and dependent theorems on the sphere, cone, and cylinder, can be reduced to sections of rectilinear solids only, and to a composition that is easily seen and readily handled.

Various ways of describing new solids.

If from a point above a plane a rigid descending straight line is moved round an area, of any shape whatever, diverse kinds of conical bodies are produced. Thus if the plane area is bounded by the circumference of a circle, a right or scalene cone is produced. Also if the figure used for the base, or the plane area, has a center—an ellipse for example—then we get an elliptic cone, which is a right cone if the given point is directly above the center, and if not it is scalene. Another conic gives another elliptic cone.

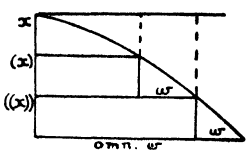
If the rigid line drawn down from the point is circular or some other curve, at one time it is so fixed to the point or pole that it has freedom to move in one way only, say round an axis, in which case it is necessary that the base should be a circle and that the fixed point or pole should be directly over the center. At another time it is necessary that the rigid line should have freedom for other motions, such as an up and down motion, or some other motion,

controlled by some straight line; and then it will always ascend or descend when necessary, so that it ever touches the given plane area by its rotation round the axis; and this is the second class of cones. A third class consists of those in which, besides the double motion of a rotation round an axis and an up and down motion, the curve alone, or the axis alone, or even both the curve and the axis, also perform other motions meanwhile, or even the point itself moves.

Here is another consideration.

The moments of the differences about a straight line perpendicular to the axis are equal to the complement of the sum of the terms; and the moments of the terms are equal to the complement of the sum of the sums, i. e.,

$$\text{omn.}\overline{xw} \sqcap \text{ult.}x,\ \overline{\text{omn.}w},, - \overline{\text{omn. omn.}w} \qquad (14)$$



Let $xw \sqcap az$, then $w \sqcap \frac{az}{x}$, and we have

$$\text{omn.}az \sqcap \text{ult.}x,\ \text{omn.}\frac{az}{x} - \text{omn.omn.}\frac{az}{x};$$

hence $$\text{omn.}\frac{az}{x} \sqcap \text{ult.}x\ \text{omn.}\frac{az}{x^2} - \text{omn.omn.}\frac{az}{x^2};$$

inserting this value in the preceding equation, we have

$$\text{omn.}az \sqcap \text{ult.}x^2\ \text{omn.}\frac{az}{x^2} - \text{ult.}x,\ \overline{\overline{\text{omn.omn.}\frac{az}{x^2}}},$$

[14] I have given this equation, and those that immediately follow it, in facsimile, in order to bring out the necessity that drove Leibniz to simplify the notation.

We have here a very important bit of work. Arguing in the first instance from a single figure, Leibniz gives two general theorems in the form of moment theorems. The first is obvious on completing the rectangle in his diagram, and this is the one to which the given equation applies. In the other the whole, of which the two parts are the complements, is the moment of the completed rectangle; its equivalent is the equation

$$\text{omn.}xy = \text{ult.}x\ \text{omn.}y - \text{omn. omn.}y.$$

Now, although Leibniz does not give this equation, it is evident that he recognized the analogy between this and the one that is given; for he immediately accepts the relation as a general *analytical theorem that he can use without any reference to any figure whatever,* and proceeds to develop it further. This would therefore seem to be the point of departure that led to the Leibnizian calculus.

$$\overline{-\,\text{omn. ult.}x,\ \textit{omn.}\frac{az}{x^2} - \text{omn.omn.}\frac{az}{x^2}}\ ;$$

and this can proceed in this manner indefinitely.

Again, $\quad \text{omn.}\,\dfrac{a}{x} \sqcap x\,\text{omn.}\,\dfrac{a}{x^2} - \text{omn.omn.}\,\dfrac{a}{x^2}\,,$

and $\quad \text{omn.}a \sqcap \text{ult.}x\,\text{omn.}\,\dfrac{a}{x} - \text{omn. omn.}\,\dfrac{a}{x}\,;$

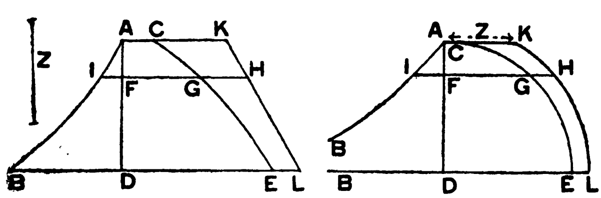
the last theorem expresses the sum of logarithms in terms of the known quadrature of the hyperbola.[15]

The numbers that represent the abscissae I usually call ordinals, because they express the order of the terms or ordinates. If to the square of any ordinate of a figure whose quadrature can be found, you add the square of a constant, the roots of the sum of the two squares will represent the curve of the quadratrix. Now if these roots of the sum of the two squares can also give an area that has a known quadrature, then also the curve can be rectified.[16]

[15] Having freed the matter from any reference to figures, he is able to take any value he pleases for the letters. He supposes that $z = 1$, and thus obtains the last pair of equations. He then considers x and w as the abscissa and ordinate of the rectangular hyperbola $xw = a$ (constant); hence omn.a/x or omn. w is the area under the hyperbola between two given ordinates, and therefore a logarithm; and thus omn. omn.a/x is the sum of logarithms, as he states.

[16] There only seem to be two possible sources for this paragraph, (1) original work on the part of Leibniz, and (2) from Barrow. For we know that Neil's methods was that of Wallis, and the method of Van Huraet used an ordinate that was proportional to the quotient of the normal by the ordinate in the original curve.

Now Barrow, in Lect. XII, § 20, has the following: "Take as you may any right-angled trapezial area (of which you have sufficient knowledge), bounded *by two parallel straight lines* AK, DL, *a straight line* AD, *and any line* KL whatever; to this let another such area be so related that when any straight line FH is drawn parallel to DL, cutting the lines AD, CE, KL in the points F, G, H, and *some determinate line* Z *is taken, the square on* FH *is equal to the squares* on FG and Z. Moreover, let the curve AIB be such that,



if the straight line GFI is produced to meet it, *the rectangle contained by* Z *and* FI *is equal to the space* AFGC; *then the rectangle contained by* Z *and the curve* AB *is equal to the space* ADLK. The method is just the same, even if the straight line AK is supposed to be infinite.

This striking resemblance, backed by the fact that there seems to be no connection between this theorem and the rest of the paper, that Leibniz gives

To describe a curve to represent a given progression.

From the square of a term of the progression, take away the square of a constant quantity; if the figure that is the quadratrix of the roots formed from the two squares is described, it will give the curve required; it does not follow that a rectifiable curve can be described.

The elements of the curve described can be expressed in many different ways. Different methods of expressing the elements of a curve may be compared with different methods of expressing a figure having similar parts with it, according as it is referred in different ways. Lastly, a solid having similar parts with a curve can thus far be expressed in many ways, and so also for a surface or figure having similar parts with the curve.

§ VI.

Three days later, Leibniz considers the possibility of being able to find the quadratrix in all cases, or when that is impossible, some curve which will serve for the quadratrix very approximately. He makes an examination of the difficulties that are likely to be met with and the means to overcome them, and he seems to be satisfied that the method can be made to do in all cases. But in the absence of an example of the method he proposes to adopt, he seems only to have been wasting his time. But this may be dismissed, for it is not here that the importance of this essay lies; it is altogether in what follows.

The rest of the essay is in the form of disjointed notes: it is just the kind of thing that any one would write *as notes while reading the works of others.* This is what I take it to be; and the works he is considering are those of

no attempt at a proof, (indeed I very much doubt whether I could have made out his meaning from the original unless I had recognized Barrow's theorem) and that Leibniz gives 1675 as the date of his reading Barrow, almost forces one to conclude that this is a note on a theorem (together with an original deduction therefrom by himself) which Leibniz has come across in a book that is lying before him, and that that book is Barrow's. Against it, we have the facts of the use of the word "quadratrix," not in the sense that Barrow uses it, namely as a special curve connected with the circle; that the quadratrix is one of the special curves that Barrow considers in the five examples he gives of the Differential Triangle method; and that another example of this method is the differentiation of a trigonometrical function which seems to be unknown to Leibniz.

Descartes, Sluse, Gregory St. Vincent, James Gregory and Barrow. Descartes he has already dismissed as impracticable in the manuscript of January, 1675; but there are indications that the former's method has still some influence. An incidental remark leads to the consideration of the *ductus* of Gregory St. Vincent; but these too are soon cast aside, truly because Leibniz does not quite grasp the exact meaning of Gregory. He then either remembers what he has seen in Barrow or refers to it again, for the next thing he gives is some work in connection with which he draws the characteristic triangle, *which is here for the first time, as far as* these manuscripts *go, the Barrow form and not the Pascal form.* He immediately obtains something important, namely,

$$\frac{\overline{\text{omn. } l^2}}{2} = \text{omn. } \overline{\text{omn. } l} \frac{l}{a}.$$

Noting that, in modern notation, l is dy, and a is dx, and also, since a is also supposed to be unity, that the final summation on the right-hand side is performed by "applying the successive values to the axis of x, while the summation denoted by omn.l is a straightforward summation, it follows that the equivalent of the result obtained by Leibniz is $\frac{1}{2}y^2 = \int y \frac{dy}{dx} dx$.

However, in attempting to put this theorem into words as a general theorem he makes an error; he quotes $\overline{\text{omn.} l^2}$ as the "sum of the squares" instead of the "square of the final y." This I think is simply a slip on the part of Leibniz, and not, as suggested by Gerhardt and Weissenborn, an indication that Leibniz confused $\overline{\text{omn.} l^2}$ with $\overline{\text{omn.} l}^2$, and considered them as equivalent. Neither of these authorities appears to have noticed the fact that when Leibniz has invented the sign $\int$ (which he immediately proceeds to do) he carefully makes the distinction between the

equivalents to the square of a sum and the sum of the squares. Thus we find that his equation is written as

$$\int \overline{\frac{l^2}{2}} = \int \overline{\int l \, \frac{l}{a}}, \qquad \text{(note the vinculum)}$$

while later in the essay we have $\int l^3$ to stand for the sum of the cubes. Further, apart from this. I do not think that any one can impute such confusion of ideas to Leibniz, if it is noted that so far this is not the differential calculus, but the calculus of differences, i. e., l is still a very small but finite line and not an infinitesimal; for in § IV, Leibniz had squared a trinomial successfully, and must have known that the sum of the squares could not be equal to the square of the sum. Both these above-named authorities seem to find some difficulty over the introduction of the letter a, apparently haphazard. This difficulty becomes non-existent, if it is remembered that a is taken to be unity, and the remarks made about dimensions by Leibniz are carefully considered; it will then be found that the a is introduced to keep the equations homogeneous! Weissenborn also remarks that Leibniz jots down the integral of x^2 without giving a proof, and appears to be in doubt how he reached it. If this is so, it confirms the opinion that I have already formed, namely, that neither Gerhardt nor Weissenborn tried to get to the bottom of these manuscripts, being content with simply "skimming the cream."

I suggest that Barrow, Gregory St. Vincent, and even Sluse, now join Descartes on the shelf or the floor, and that the rest of the essay is all Leibniz. He writes the two equations he has found, the equivalents to two theorems obtained geometrically, notes the fact that these are true for infinitely small differences (without, however, mentioning that they are *only* true in such a case), discards diagrams, and proceeds analytically; that is, the y's are successive values of some function of x, where the values

of x are in arithmetical progression; hence, substituting x for l in the equation

$$\text{omn}.xl = \text{omn}.l - \text{omn}.\text{omn}.l,$$

and remembering that $\text{omn}.x = x^2/2$, as he has proved, we have

$$\text{omn}.x^2 = x\frac{x^2}{2} - \text{omn}.\frac{x^2}{2}, \text{ or omn}.x^2 = \frac{x^3}{3}.$$

Again, below he gives $\int \frac{x^3}{3} = \frac{x^4}{4}$ correctly (although there is an obvious slip or, as I think, a misprint of l for x); this could have been obtained in the same way.

$$\text{omn}.x^3 = x\frac{x^3}{3} - \text{omn}.\frac{x^3}{3}, \text{ or omn}.x^3 = \frac{x^4}{4}.$$

Similarly, Leibniz could have gone on indefinitely, and thus obtained the integrals of all the powers of x. But his brain is too active; as Weissenborn says, his soul is in the throes of creation. He merely alludes in passing to the inverse operation to $\int$ as being represented by d, which he for some reason writes in the denominator (probably erroneously because he has noted that $\int$ increases the dimensions); and then he harks back to the opening idea of the essay, the obtaining of the quadratrix by means of transformation of equations, an idea truly as hopeless as the method of Descartes which he has discarded. Nevertheless, even then he obtains something remarkable, nothing more or less than the inverse of the differentiation of a product. This fundamental theorem is obtained geometrically; the proof of the little theorem on which the final result is founded is not given, neither is there a diagram. It cannot therefore be supposed but that Leibniz is working from a diagram already drawn, and I suggest he was referring to one of those theorems, with which he had filled "hundreds of pages" between 1673 and 1675. The

proof follows quite easily by the use of the characteristic triangle, and is given in a footnote. This theorem is not in Barrow, nor can I remember seeing it in Cavalieri; I have not yet been able to procure a Gregory St. Vincent; it may be in James Gregory.

The benefits of this discovery are lost as before, for Leibniz once more alludes to the transformation of equations for the purpose of obtaining the quadratrix.

Summing the whole essay, we can say that in it is the beginning of the Leibnizian *analytical calculus.*

29 October, 1675.

Analyseos Tetragonisticae pars secunda.

(Second part of analytical quadrature.)

I think that now at last we can give a method, by which the analytical quadratrix may be found for any analytical figure, whenever that is possible; and, when it can not be done, it will yet always be possible that an analytical figure may be described, which will act as the quadratrix as nearly as is required. This is how I look at it:

Suppose the equation of the curve, of which the quadratrix is required, is given, and that the unknowns in it are x and v. Let the equation to the curve required be[17]

$$v = b + cx + dy + ex^2 + fy^2 + gyx + hy^3 + lx^3 + mxyy + yxx + \text{etc.}; \quad \ldots \quad \text{(i)}$$

let it be set in order for tangents, as follows:

$$-dy - 2fy^2 - gyx - 3hy^3 - 2mxy^2 - mx^2y - \text{etc.}$$
$$= ct + 2ext + gyt + 3lx^2t + my^2t + 2yxt + \text{etc.} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

[17] This is either a misprint, v instead of O, or else Leibniz is in error. For Slusius's method there must be only two variables in the equation. In the *Phil. Trans.* for 1672 (No. 90), Sluse gives his method thus:

If $y^5 + by^4 = 2qqv^3 - yyv^3$, then the equation must be written $y^5 + by^4 + yy^3 = 2qqv^3 - yyv^3$; then multiply each term on the left-hand side by the number of y's in the term, and substitute t in place of one y in each; similarly multiply each term on the left-hand side by the exponent of v; the equation obtained will give the value of t.

The use of the letters v and y is to be noted in connection with Leibniz's use of the same letters; it does not seem at all necessary that Leibniz should have seen Newton's work, with this ready to the former's hand, as a member of the Royal Society. I suggest that Sluse obtained his rule by the use of a and e, as given in Barrow. Can Barrow's words *usitatum a nobis* (in the midst of a passage written in the first person singular) have meant that the method was common property to himself ***and several other mathematicians that were contemporary with him***? This would explain a great deal.

Now, $t/y = a/v$; hence, if from the equation $t/y = a/v$, we eliminate t and y by the help of equations (i) and (ii), that equation should be produced which represents the figure that has to be quadratured; and by comparing the terms of the equation thus obtained with the given equation, unless indeed there is no possibility of comparing them, we shall obtain the quadrature.

But if an impossibility arises, it is then known that the given analytical figure has no analytical quadratrix. But it is quite clear that if we add to it such as will change it almost imperceptibly, then a quadrible figure may be obtained, since this plainly produces another equation. However, as an impossible case may arise, we must consider the difficulties.

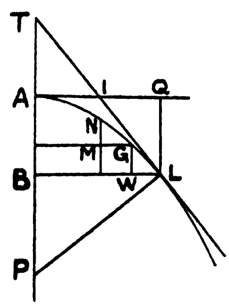
Say that the equation that is obtained is of infinite prolixity, while the given one is finite. My answer is, that in comparing the one with the other it will be seen how far at most the powers of the unknowns in the indefinite equation can go. The retort may be made, that it may happen that the indefinite equation obtained may have more terms than the finite equation that is given and yet may be reduced to it, for it may be divided by something else that is either finite or indefinite. This difficulty hindered me for a long time a year ago, but now I see that we should not be stopped by it. For it may happen that from a certain determinate figure (whose equation is not divisible by a rational) by the method of tangents there may arise an ambiguous figure; for it is impossible to say that, for *any* figure, there shall be only one tangent at any one point. Hence the produced equation can neither be divided by a finite nor by an indefinite quantity; for in truth indefinite figures, or those whose ordinates are represented by an infinite equation, have sometimes these very ordinates finite, and these ought to satisfy the equation. Notwithstanding that, I foresee another difficulty; for indeed it seems that sometimes it may happen that all the roots of the equation will not serve for the solution of the problem. Yet, to tell the truth, I believe they will do so.

Now here is a difficulty that really is great. It may happen that a finite equation may also be expressed as an indefinite one, so that the equation obtained may really be the same as the given equation although it does not appear to be. For example,

$$y^2 = x/(1+x) = x - x^2 + x^3 - x^4 + x^5 - x^6 + \text{etc.};$$

and in the same way others can be formed by various compositions and divisions. This I confess is truly a difficult point, but it can be

answered thus: If a figure has an analytical quadratrix of any sort, in all cases it may be assumed to be an indefinite one; and then it will not in all cases give an indefinite, but sometimes a finite, equation that is equivalent to the given equation. In the same way, it is certain that the quadratrix of a given curve as it is usually investigated, whenever there is one, will also be determined; and that too given uniquely and not ambiguously, so that any that differs from it, differs only in name. There is still one difficulty left; it seems impossible to determine which is the end or first term of the indefinite equation that is obtained; for it may happen that the terms of lower degree are cut out, and then it is divisible by y or x or yx or powers of these; nor do I see that there is anything to prevent this. There is the same difficulty whether you start from the lowest or the highest degree in the equation assumed to begin with as indefinite. Suppose then that in the equation obtained this



division is possible, then it is necessary that the constant term should be absent, and also all those terms in which x alone or, if you like, all the terms in which y alone is absent; and if we examine this continuously we may light upon an impossibility.

In this general calculus then, we may take it as certain that this difficulty is solved, and that such a division after the calculation can never happen; or if it is possible for it to happen, then the terms will go out, one after the other, so that the equation can be depressed and the comparison be made; and then it is to be seen whether this difficulty cannot be overcome in general, and the comparison proceed as we proceed with the elimination. Perhaps if the figure to be quadratured is reduced beforehand to its simplest equation possible, impossibilities will the more readily be detected. For then presumably the quadratrix must become more simplified. In addition we have another source of assistance; for various cal-

culations leading to the same thing, though obviously differing from one another, can be contrived, from which equations are comparable.

Let $BL = y$, $WL = l$, $BP = p$, $TB = t$, $GW = a$, then $y = \text{omn.}l$.

Incidentally I may remark that there are composite numbers that cannot be added or subtracted from one another by parts, namely those denominated by powers, or by sub-powers or surds. There are also other denominate numbers which cannot be multiplied together by parts, such as numbers representing sums; for instance, omn.l cannot be multiplied by omn.p, nor can we have $y^3 = 2\text{omn. omn.}pl$. However, as such a multiplication may be imagined to occur under certain conditions, we must consider it as follows:

We require the space that represents the product of all the p's into all the l's; we cannot make use of the ductions of Gregory St. Vincent, where figures are multiplied by figures, for by this method one ordinate is not multiplied by all the others, but one into one. You may say that if one ordinate is multiplied by all the rest it will produce a sursolid space, namely, the sum of an infinite number of solids. For this difficulty I have found a remedy that is really admirable. Let every l be represented by an infinitely short straight line WL, that is, we want the quadratrix line representing omn. l; well, the line $BL = \text{omn. } l$; and if this is multiplied by every p, each represented by a plane figure, then a solid is produced. If all the l's are straight lines and all the p's are curves, a curved surface is produced by a duction of the same sort. But these things are all old; now, here is something new.

If upon WL, MG, or every single l, is superimposed the same curve representing all the p's, where the curve p is originally all in the same plane and is carried along the curve AGL while its plane always moves parallel to itself, then what we require will be obtained. In place of a curve a plane may be carried along the curve in the same manner, and a solid will be obtained, whereas by the former method it was a curvilinear surface; and both for the surface and for the solid the section always remains the same. It remains to be seen whether a number of analytical surfaces cannot be ascertained, as in the case of analytical lines; but this is mentioned only incidentally.

N. B. The curvilinear surface formed by the motion of a curve parallel to itself along the curve will be equal to the cylinder

of the curve under BL, the sum of all the l's but this is also mentioned incidentally.

To resume, $\frac{l}{a} = \frac{p}{\text{omn.}\, l} = y$, therefore $p = \frac{\overline{\text{omn.}\, l}}{a}\, l$. Hence, omn. $y\ \frac{l}{a}$ does not mean the same thing as omn.y into omn.l, nor yet y into omn.l; for, since $p = \frac{y}{a}\, l$ or $\frac{\overline{\text{omn.}\, l}}{a}\, l$, it means the same thing as omn.l multiplied by that one l that corresponds with a certain p; hence, omn.p = omn.$\frac{\overline{\text{omn.}\, l}}{\text{a}}\, l$. Now I have otherwise proved omn.$p = \frac{y^2}{2}$, i. e., $= \frac{\overline{\text{omn.}\, l^2}}{2}$; therefore we have a theorem that to me seems admirable, and one that will be of great service to this new calculus, namely,

$$\frac{\overline{\text{omn.}\, l^2}}{2} = \text{omn.}\ \overline{\overline{\text{omn.}\, l}\,\frac{l}{a}}, \text{ whatever } l \text{ may be;}$$

that is, if all the l's are multiplied by their last, and so on as often as it can be done, the sum of all these products will be equal to half the sum of the squares, of which the sides are the sum of the l's or all the l's. This is a very fine theorem, and one that is not at all obvious.

Another theorem of the same kind is:

$$\text{omn.}xl = x\,\text{omn.}l - \text{omn.omn.}l,$$

where l is taken to be a term of a progression, and x is the number which expresses the position or order of the l corresponding to it; or x is the ordinal number and l is the ordered thing.

N. B. In these calculations a law governing things of the same kind can be noted; for, if omn. is prefixed to a number or ratio, or to something indefinitely small, then a line is produced, also if to a line, then a surface, or if to a surface, then a solid; and so on to infinity for higher dimensions.

It will be useful to write $\int$ for omn., so that

$$\int l = \text{omn.}l, \text{ or the sum of the } l\text{'s.}$$

Thus, $$\int \frac{\overline{l^2}}{2} = \int \overline{\int \overline{l}\, \frac{l}{a}}, \text{ and } \int \overline{xl} = x \int \overline{l} - \int \int l.$$

From this it will appear that a law of things of the same kind

should always be noted, as it is useful in obviating errors of calculation.

N. B. If $\int l$ is given analytically, then l is also given; therefore if $\int\int l$ is given, so also is l; but if l is given, $\int l$ is not given as well. In all cases $\int x = x^2/2$.

N. B. All these theorems are true for series in which the differences of the terms bear to the terms themselves a ratio that is less than any assignable quantity.

$$\int x2 = \frac{x^3}{3}$$

Now note that if the terms are affected, the sum is also affected in the same way, such being a general rule; for example, $\overline{\int \frac{a}{b} l} = \frac{a}{b} \times \overline{\int l}$, that is to say, if $\frac{a}{b}$ is a constant term, it is to be multiplied by the maximum ordinal; but if it is not a constant term, then it is impossible to deal with it, unless it can be reduced to terms in l, or whenever it can be reduced to a common quantity, such as an ordinal.

N. B. As often as in the tetragonistic equation, only one letter, say l, varies, it can be considered to be a constant term, and $\int l$ will equal x. Also on this fundamental there depends the theorem:

$$\int \frac{l^2}{2} = \int \overline{\int \overline{l}\, l}, \text{ that is, } \frac{x^2}{2} = \int x.$$

Hence, in the same way we can immediately solve innumerable things like this; thus, we require to know what e is, where

$$\int \overline{\frac{c}{a} \int \overline{l} + ba^2} + \int l^3 \overset{(18)}{+} \int l^3 = ea^3;$$

we have

$$a^3 e = \frac{cx^3}{3} + ba^2 x + \frac{x^4}{4} + xa^3.$$

For indeed $\int l^3 = x$, because l is supposed to be equal[19] to a for the purpose of the calculation; $\int \frac{l}{a} = x$.

[18] There is evidently a slip here; l should be x.

[19] This is an instance of the care which Leibniz takes; in the work above l has been the difference for y, and a the difference for x; he is now integrating an algebraical expression, and not considering a figure at all; hence $l = a$, and a is equal to unity, and therefore $\int l^3 = l^3 x = a^3 x = x$! Thus what is generally considered to be a muddle turns out to be quite correct. The muddle is not with Leibniz, it is with the transcriber. It is certain that these manuscripts want careful republishing from the originals; won't some millionaire pay to have them reproduced photographically in an *edition de luxe*?

Also $\int \overline{c \int \overline{l^2}} = \frac{cx^3}{3}$, that is $= \frac{c \int \overline{l^3}}{3a^3}, \int ba^2 = \int l\ ba.$

Also it is understood that a is unity. These are sufficiently new and notable, since they will lead to a new calculus.

I propose to return to former considerations.

Given l, and its relation to x, to find $\int l$.

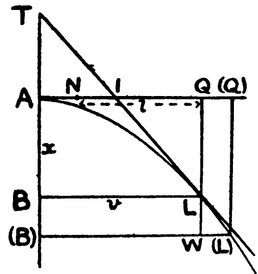
This is to be obtained from the contrary calculus, that is to say, suppose that $\int l = ya$. Let $l = ya/d$; then just as $\int$ will increase, so d will diminish the dimensions. But $\int$ means a sum, and d a difference. From the given y, we can always find y/d or l, that is, the difference of the y's. Hence one equation may be transformed into the other; just as from the equation $\int \overline{c \int \overline{l^2}} = \frac{c \int \overline{l^3}}{3a^3}$, we can obtain the equation $c \int \overline{l^2} = \frac{c \int \overline{l^3}}{3a^3 d}$.

N. B. $\int \frac{x^3}{b} + \int \frac{x^2 a}{e} = \int \overline{\frac{x^3}{b} + \frac{x^2 a}{e}}$. And in the same manner, $\frac{x^3}{db} + \frac{x^2 a}{de} = \frac{\frac{x^3}{b} + \frac{x^2 a}{e}}{d}$.

But to return to what has been done above. We can investigate $\int l$ in two ways; one, by summing y and seeking $\underline{ya/d = 1}$; the other, by summing $z^2/2a = y$, or by summing $\sqrt{2ay} = z$, and then $z^2/t = p = l = ya/d$. Hence, if in an indefinite equation, we eliminate y by substituting in its place $z^2/2a$, and investigate the t of this new equation which is indefinite like the first, and then by the help of the value $z^2/t = l$, and after that by the help of the new value of t, eliminate z from the indefinite equation containing z and t, there will remain out of the (three) letters x, z, t, l, the letter l alone; and again we ought to get an equation which should be the same not only as the given one, but also the same as the one that was obtained a little while ago. Hence, since we have two indefinite equations, containing not only the principle quantities, but also arbitrary ones, yet not altogether unlike the former; and these ought to be identical; it will appear to show whether certain terms cannot be eliminated, whether it is not possible that a comparison should be made, and other things of the sort; and, what is really the most important thing, which terms are really the greatest and the least, or the number of terms of the equation.

Moreover, since in the similar triangles TBL, GWL, LBP, no

mention has yet been made of the abscissa x or of the fixed point A, let us then suppose that through the fixed point A there is drawn an unlimited straight line AIQ, parallel to LB, meeting the tangent LT in I; and let AQ = BL; bisect AI in N; then I say that the sum of every QN will always be equal to the triangle ABL, as can easily be shown by what I have said in another place.[20]



These considerations give once more a fresh fundamental theorem for the calculus. For $xv/2 = y$, where we suppose that BL $= v$ and QN $= l$, and $y = \int l$;

but $\dfrac{\text{AI}}{v} = \dfrac{t-x}{t}$, therefore AI $= \dfrac{t-x}{t}v$,

and QI $= v - \text{AI} = v\dfrac{tt}{t}v + \dfrac{xv}{2t}$, i. e. QI $= \dfrac{xv}{t}$,

$$\therefore \text{QN} = \text{QI} + \frac{\text{AI}}{2} = \frac{xv}{t} + \frac{v}{2} - \frac{xv}{2t} = \frac{xv + tv}{2t} = l. \qquad (21)$$

Now, by the help of the equation $(xv + tv)/2t = l$, and of the former equation $y = xv/2$, and taking once more the first indefinite or general equation as a third, and eliminating first of all y, then t by means of the value found for the ratio of t to x from the indefinite equation containing x and v, and lastly v by the help of the equation $(xv + tv)/2t = l$, in which the principal quantities x and l alone remain, as before; and this again should be identical with the given equation.

Thus we have found three equations obtained in different ways, which should all be identical with one another and with the given equation; and these three are not only identical but should also

[20] Since the triangles QLI, WL(L) are similar, QI.B(B) = QL.Q(Q), hence omn.QI (applied to AB) = omn. QL (to AQ) = figure AQLA, hence omn.(QI + QA) = rect. ABLQ = 2△ABL.

[21] Since l is the difference for y, therefore $2l$ is the difference for xv; this is shown to be $(xv + tv)/t$ or $x(v/t) + v$; and this is the equivalent to (since $v/t = dv/dx = dv$)

$$d(xv) = xdv + v = xdv + vdx.$$

consist of the same letters and signs; and whether this is possible, will immediately appear on being worked out analytically.

§ VII.

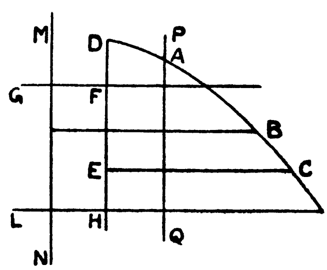
The next manuscript is a further continuation of the preceding, written two days later. In this Leibniz returns to the idea that he has found so prolific, namely, the moments of a figure. It is to be observed that he speaks of the method of breaking up an area into segments as something that he has already worked out; this will be remarked upon in a note on a later manuscript, where it will help to clear up a small difficulty. The accuracy of the rather involved algebraical work is also a point to be noticed.

1 November, 1675.

Analyseos Tetragonisticae pars tertia.

(Third part of Analytical Quadrature.)

It was some time ago that I observed that, being given the moment of a curve ABC, or of a curvilinear figure DABCE, about two straight lines parallel to one another, such as GF, LH (or MN,



PQ), then the area of the figure could be obtained; because the two moments differed from one another by the cylinder of the figure, where the altitude was the distance between the parallels.

Now, this is true of every progression, whether of numbers or of lines; that is, even if we do not use curvilinear figures but ordinated polygons; in other words, where the differences between the terms are not infinitely small. Suppose we have any such ordinated quantity z, and let the ordinal number be x, then

$$b\,\overline{\text{omn.}z} \sqcap \pm\,\text{omn.}\overline{zx} \mp \text{omn.}\overline{zx+b}$$

and this is evident by the calculus alone.

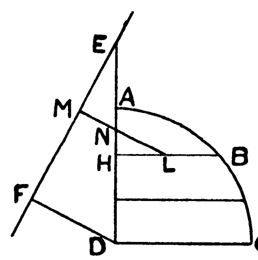
By the help of this rule, the sums of terms of an arithmetical

progression refolded reciprocally;[22] and this multiplication takes place when it is required to find the moment of the ordinates about a straight line perpendicular to the axis. But if the moment about any other straight line is required, there is the following general rule:

From the center of gravity of each of the quantities of which the moment is required, a perpendicular is drawn to the axis of libration; then the sum of the rectangles contained by the distances or perpendiculars and the quantities will be equal to the moment about the given straight line.

Hence, if the given straight line is the axis of equilibrium, it immediately follows that the moment of the figure about the axis is equal to the sum of the half-squares. Also when it is parallel to that, it will differ from the foregoing by a known quantity.

Now, let us take another straight line: for the circle for instance, let ABCD be a quadrant, vertex B, and center D; let another straight line be given, that is to say, let the prependicular DF be given and



also EF where it meets the diameter, and thus also DE; let HB be the general ordinate to the circle, and L its middle point; let LM be drawn perpendicular to EF.

Then it is clear that the triangles EFD, EMN (where N is the intersection of ML and AD), and LHN are similar.

Let AD$=x$, then HL $= \frac{y}{2} = \frac{\sqrt{a^2 - x^2}}{2}$. But, on account of the similar triangles, $\frac{\mathrm{NH}}{\mathrm{HL}} = \frac{\mathrm{DF}(=d)}{\mathrm{FE}(=f)}$;

therefore

$$\mathrm{NH} = \frac{d}{2f}\sqrt{a^2 - x^2} = \frac{yd}{2f}.$$

[22] The meaning of this is probably a series such as that considered by Wallis. If a, $a+d$, $a+2d$, etc. is the arithmetical progression, and l, $l-d$, $l-2d$, etc. is the series reversed, then the series refolded reciprocally is al, $(a+d)(l-d)$, $(a+2d)(l-2d)$, etc. It may however mean the sum of the squares of the arithmetical progression. But the point is not very important.

Hence, $EN = DE(=e) - HD(=x) - NH\left(=\frac{yd}{2f}\right) = e - x - \frac{yd}{2f}$.

Now $NL = \sqrt{NH^2 + HL^2} = \sqrt{\frac{d^2}{4f^2}y^2 + \frac{y^2}{4}} = \frac{y}{2}\sqrt{\frac{d^2}{f^2}+1}$;

and $\frac{MN}{EN} = \frac{NH}{HL}$, or $MN = \frac{NH.EN}{HL}$; thus we have

$$MN = \frac{dy}{2f}\frac{\overline{e-x-\frac{dy}{2f}}}{\frac{y}{2}\sqrt{\frac{d^2}{f^2}+1}} = \frac{d}{f\sqrt{\frac{d^2}{f^2}+1}}\overline{e-x-\frac{dy}{2f}};$$

and $ML = MN + NL = \frac{d}{f\sqrt{\frac{d^2}{f^2}+1}}\overline{e-x-\frac{dy}{2f}} + \frac{y}{2}\sqrt{\frac{d^2}{f^2}+1}$;

hence, since $e = \sqrt{f^2-d^2}$, we have[23]

$$ML = \frac{d\overline{\sqrt{f^2-d^2}-x-\frac{d}{2f}y}+\frac{d^2+f^2}{2f}y}{\sqrt{d^2+f^2}} = \frac{d\overline{\sqrt{f^2-d^2}-x}+\frac{fy}{2}}{\sqrt{d^2+f^2}}$$

and this calculation is general for any curve, so long as x is always taken as the abscissa and y as the ordinate.

Therefore the rectangle contained by ML and HB $(=y)$, or the moment of each ordinate taken with regard to the straight line EF, or *wa*, will be equal to

$$\frac{d\overline{\sqrt{f^2-d^2}y-xy}+\frac{f}{2}y^2}{\sqrt{f^2+d^2}}$$

Hence, omn.w will be obtained from the known values of omn.x, omn.xy, and omn.y^2; also, if any three of these four are given, the fourth is also known.

Now, omn.xy will be equal to the moment of the figure about the vertex, omn. y^2 will be equal to the moment of the figure about the axis; hence, given three moments of the figure, that is to say, the moments about two straight lines at right angles and any third, the area is given.

This theorem, however, is less general than the one that was given before, in the first part of this essay, where it does not matter

[23] The accuracy of the algebra is noteworthy in comparison with the inaccuracies that occur later. There is however a slip: $e^2 = f^2 + d^2$ and not $f^2 - d^2$; this must be a slip and not a misprint, because it persists throughout. It should be noted that the figure given by Gerhardt is careless in that LM is made to pass through A.

what the angle between the straight lines may be, if only we are given three moments; but it is always understood that they are in the same plane. (Meanwhile, however, this theorem will suffice for the curve of the primary hyperbola; for, if f is infinite, or if FE and ED are parallel, $dy + y^2/2 = wa$, as has already been proved.)

It is to be observed that by other calculation the area of a quantity, whose center of gravity lies in a given plane (even though the whole quantity does not), can be found from three given moments about three straight lines in that plane. From this it is to be seen whether the results obtained, when compared with one another, will not produce something new.

If instead of the moment of a figure we require the moment of all the arcs BP, PC, etc., the perpendiculars are to be drawn from the points B, P, C, etc. only, to the straight line; for it will make no difference whether they are drawn from the end or from the middle of BP, for instance, for the difference between two such perpendiculars is infinitely small. Hence, calling the element of the curve z, the moment of the curve about the straight line EF is

$$\frac{d\sqrt{f^2-d^2}z - dxz + fyz}{\sqrt{d^2+f^2}}$$

Most of the theorems of the geometry of indivisibles which are to be found in the works of Cavalieri, Vincent, Wallis, Gregory and Barrow, are immediately evident from the calculus; as, for instance, that the perpendiculars to the axis are equal to the surface or moment of the curve about the axis, for you find that a perpendicular is equal to the rectangle contained by an element of the curve and the ordinate. Therefore I do not set any value on such theorems, or on those about applications of intercepts on the axis (intercepted between the tangents and the ordinates) to the base. Such theorems bring forth nothing new, except maybe they afford formulas for the calculus.

But my theorem about the dimensions of the segments does bring out a new thing, because the space whose dimension is sought is broken up in a different way, that is to say, not only into ordinates but into triangles. Also perhaps the Centrobaric method yields something new. Maybe an easy method can be obtained, by which without diagrams those things which depend on a figure can be derived by calculus. Gregory's theorem, on ductions of two

parabolas,[24] one under the other, equal to a cylinder, is immediately evident by calculus; for the ordinate of a circle $y=\sqrt{a^2-x^2}$, that is, the product of $\sqrt{a+x}$ and $\sqrt{a-x}$; and in the same way, $\sqrt{2av-v^2}=y$, which gives $y=\sqrt{v}$ into $\sqrt{2a-v}$; and these come to the same thing.

If the same ordinate y is multiplied by some quantity z, and afterward by the same $z\pm$ some known or constant number b, the difference between the sums produced will be equal to the cylinder of the figure; so that

$$zy_{,,}-zy+by \sqcap by.$$

Although this is evident in general by itself, yet applications of it are not always evident. For instance, let

$$y=\frac{x^2}{ax-b^2}=\frac{x^2}{\sqrt{ax}+b,\ \sqrt{ax}-b};$$

then, multiplying by $\sqrt{ax}+b$, we have $\dfrac{x^2}{\sqrt{ax}-b}$; .... (A)

and, multiplying by $\sqrt{ax}-b$, we have $\dfrac{x^2}{\sqrt{ax}+b}$; .... (B)

but, since instead of $\dfrac{ax^2}{ax-b^2}$, we can have $x+\dfrac{b^2x}{ax-b^2}$,

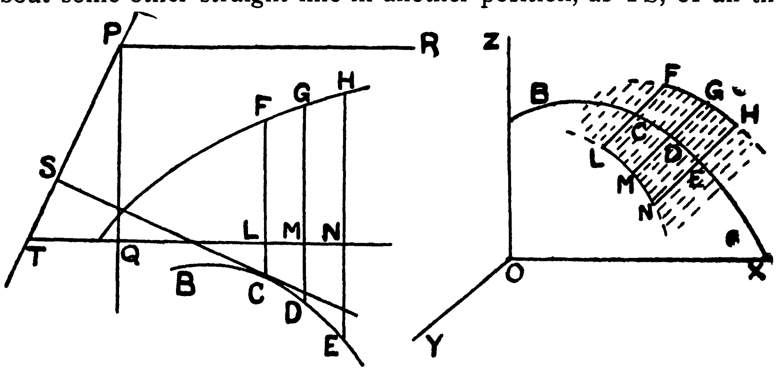
which depends on the quadrature of the hyperbola; and thus if one of the two things, (A) or (B), is given, then the other is also known, supposing that the quadrature of the hyperbola is known.

Suppose that at the points C, D, E of a curve situated in any plane there are imposed, perpendicular to the plane, the ordinates of another curve FGH (not necessarily of the same constitution), in such a manner that the middle point of each of these ordinates lies in the plane; then it is evident that LG, MD, NE, multiplied by FL, GM, HN, (that is, the lines imposed at C, D, E of the curve BCDE) or the rectangles FLG, GMD, HNE, or the duction of these two planes into one another, will be equal to the moment of every LC, MD, NE, etc. Hence, if PR is another axis, and the interval between it and QL is the straight line PQ, the moment

[24] Such theorems are also considered in Wallis, where it is shown that the products for two equal parabolas are the squares on the ordinates of a semicircle; the axes of the parabolas being coincident, but set in opposite sense.

about PR differs from that about QL by the cylinder whose base is LC, MD, etc., and whose height is PQ.[25]

But, if the moment about the straight line PQ, and also that about some other straight line in another position, as TS, of all the



ordinates LF of the same figure, imposed at the points C, then we shall have the cylinder corresponding to all the LF's, as I will now prove.

If we call QL, x, and CL, y, then TC$=\frac{f}{a}x+\frac{g}{a}y+h$; and this multiplied by z, where FL or MG$=z$, will give

$$\frac{f}{a}xz+\frac{g}{a}yz+hz.$$

Now xz is given, being the supposed moment about PQ, which is the same whether the z's are placed where they were in the lines LF, MG, etc., or at the points C, D, E. Also yz is given, either as the rectangle FLC or as the duction, by hypothesis. Hence, if in addition there is given one moment of the ordinates imposed upon

[25] This is obviously wrong; the base of the cylinder is the area made up of FL, GM, HN, etc. The whole of this last passage proved to be difficult to make out; Leibniz has not completed his figure, by showing the surface formed by placing the ordinates FL, GM, HN with their middle points at C, D, E, and the ordinates themselves perpendicular to the plane of the curve BCDE, which figure I have added on the right-hand side of Leibniz's figure. Even when this is given, there is another difficulty added because as given by Gerhardt, CS is the tangent at D instead of the proper line, namely, the perpendicular from C to TS; in addition through a misprint, this line is afterward referred to as TC. Lastly, "the rectangle FLG" is a misprint for FLC, which with Leibniz stands for FL.LC; this notation for a rectangle is, as far as I can remember, used by Wallis and Cavalieri.

When all these errors are revised, what at first sight seemed to be rather a muddle turns out to be an exceedingly neat idea in connection with the moments of a figure, and their use to find an area, although mostly impracticable.

Note. The values f, g, a, h, are the lengths of TQ, QP, PT, and the perpendicular from Q on PT.

the curve at the points C, D, E, and this is taken to be equal to $\frac{f}{a} xz + \frac{g}{a} yz + hz$, then we have hz or the cylinder required.

Hence, the curve BCDE is to be chosen such that the ordinates of the given curve can be multiplied by different ordinates of the former, drawn either to the axis QL or to the axis TS, with some advantage of simplicity; and the curves that are suitable for this are those that have several suitable axes, such as the circular or primary hyperbola, which has a pair of asymptotes, or an axis and a conjugate axis.

§ VIII.

Much comment has been made on the fact that the date of the next manuscript was originally "11 November 1675"; that the 5 had been altered to a 3, the ink being of a darker shade; and that it is almost certain that this alteration in date was made for some ulterior motive by Leibniz himself. Hence, if he was capable of falsifying a date in one particular case, then he is not to be trusted in others,. . . ., and so on. Instead of trying to explain away this alteration, let us try to find an explanation as to the reason of its having been made by Leibniz; I offer the following as at least feasible.

The essay starts with the words, "*Jam superiore anno mihi proposueram questionem,. . . .*" I suppose that by this Leibniz intended: "A year or two ago, I set myself the question,. . . . ." This conforms with what follows; the theorem that he sets down is one such as those that were suggested to him by Huygens, and further theorems that came to him as deductions during his first intercourse with Huygens. Years later, I therefore suggest, Leibniz refers to this manuscript, reads his own Latin, *superiore anno,* as "in the above year," gets no further, recognizes the theorem by its figure as one of the Huygens-time batch, and says to himself "1675? No, that's wrong, should be 1673,"

and proceeds to alter it to what he remembers was the date for the first consideration of the theorem.

N. B. Gerhardt himself has remarked on the darker tint of the ink used in the alteration; hence my argument, made at a later date.

The date 1675 is incontestable; for this composition is quite glaringly a development of the work that has been so efficiently started in that of November 1, 1675. Progress is still delayed by the idea that has obsessed Leibniz up till now, that of the transformation of equations, so as to be able to eliminate more unknowns than the original number of his equations warrant. He sets himself the problem: "To determine the curve in which the distance between the vertex and the foot of the normal is reciprocally proportional to the ordinate," i. e., the solution of the equation $x + y\,dy/dx = a^2/y$, in modern notation. This is a very unlucky choice for him: for I have it on the authority of Prof. A. R. Forsyth that this is incapable of solution in ordinary functions or even by a series in which the law of the series is easily and simply expressible—at least he confesses that he is unable to obtain such a solution, which I take it comes to the same thing.

Leibniz professes to have found the solution and gives $(y^2 + x^2)(a^2 - yx) = 2y^2 \log y$; and unfortunately this false success but enhances the value in his eyes of the method mentioned above. But from the equation given as the solution we may draw an incontestable conclusion; for in a previous problem Leibniz verifies his solution by the method of tangents, i. e., by differentiation, although the method does not as yet convey that idea to him; but he does not verify the solution in this case, *because he is unable at this date to differentiate the product* $y^2 \log y$.

The introduction of dx instead of x/d marks a further advance, more important perhaps than the use of $\int y\,dy$;

for he still writes $\int x$, considering dx to be constant and equal to unity. He is beginning to grasp the infinitesimal nature of his calculus, and that infinitesimals are not to be neglected because of their intrinsic smallness, but because of their smallness *with respect to other quantities* which come into the same equations and are finite; but he is far from being certain about it as yet, as is evidenced by the discussion as to whether $d(v/\psi) = dv/d\psi$ or not. However, the whole manuscript marks a distinct advance on anything that has gone before. From now on he probably discards geometry, and only refers to Descartes, Gregory and Barrow for examples to show how much superior is his method to theirs. I put his final reading of Barrow down to the interval between the date of this manuscript, 11 November, 1675, and November, 1676; it is at this time that he inserts his sign of integration in the margins of the theorems. The next person that examines the originals of these manuscripts (I am convinced that this is very necessary), should carefully see whether the ink used for the note "*novi dudum*" (which I have mentioned) is the same as that used for the sign of integration; also the other books that were used by Leibniz in his self-education should be searchingly scrutinized for clues.

The last remark I have to make is one of astonishment at the errors in the algebraical work which brings this essay to a close, and to a less degree throughout the essay; for we have seen the accuracy to which Leibniz has attained in a previous manuscript; of course, a great deal of erroneous work can be explained by supposing none too careful transcription; but a re-examination of the whole of the Leibnizian remains should include a careful scrutiny on the point as to whether some of the extracts given by Gerhardt are not the work of pupils of Leibniz, whose writing would naturally be somewhat similar. Perhaps too some of those early geometrical theorems might be un-

earthed; and this would well reward the most painstaking search. Nobody can assert that anything like an adequate tale of the progress of the Leibnizian genius has so far been told.

11 November, 1673.[26]

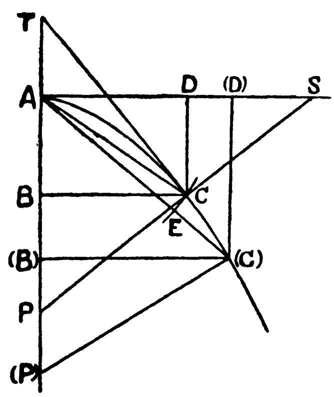
Methodi tangentium inversae exempla.

(Examples of the inverse method of tangents.)

A year or two ago I asked myself the question, what can be considered one of the most difficult things in the whole of geometry, or, in other words, what was there for which the ordinary methods had contributed nothing profitable. To-day I found the answer to it, and I now give the analysis of it.

Find the curve C(C), *in which* BP, *the interval between the ordinate* BC *and* PC *the normal to the curve, taken along the axis* AB(B), *is reciprocally proportional to the ordinate* BC.

Let AD(D) be another straight line perpendicular to the axis AB(B), and let ordinates CD be drawn to it, so that the abscissae



AD along the axis AD(D) are equal to the ordinates BC to the axis AB(B), and the ordinates CD to the axis AD(D) are equal to the abscissae AB along the axis AB(B). Let us call AD = BC = y, and AD = BC = x; also let BP = w and B(B) = z. Then it follows from what I have proved in another place that

[26] See Cantor, III, p. 183; but neither Cantor nor Gerhardt appears to offer any suggestion as to why this date should have been altered.

$$\int wz = \frac{y^2}{2}, \text{ or } wz = \frac{y^2}{2d}\text{[27]}$$

But from the quadrature of a triangle it is evident that $\frac{y^2}{2d} = y$; and therefore $wz = y$.

Now, from the hypothesis, $w = b/y$, for thus w and y will be reciprocally proportional to one another. Hence we have

$$\frac{bz}{y} = y, \text{ and thus } z = \frac{y^2}{b}.$$

But $\int z = x$, *hence* $x = \int \frac{y^2}{b}$; and from the quadrature of the parabola $\int \frac{y^2}{b} = \frac{y^3}{3ba}$; hence, $x = \frac{y^3}{3ba}$; and this is the required equation expressing the relation between the ordinates y and the abscissae x of the curve C(C), which was to be found. Therefore we consider that the curve has been found and it is analytical; in short, it is the cubical parabola whose vertex is A.

We will therefore see whether the truly remarkable theorem is not true, namely, in the cubical parabola C(C), the intervals BP between the normals to the curve, PC, and the ordinates to the axis, BC, taken along the axis ABP, are reciprocally proportional to the ordinates, BC.

The truth of this is easily shown by the calculus of tangents. For the equation to the cubical parabola is $xc^2 = y^3$; taking c to be the *latus rectum*, and supposing that for c^2 we put $3ba$, or $c = \sqrt{3ba}$, we have $3xba = y^3$.

Now, by Slusius's method of tangents, we have $t = y^3/3ba$, where t is put for BT, the interval along the axis between the tangent and the ordinate.

But $BP = w = \frac{y^2}{t}$, and therefore $w = \frac{\frac{y^2}{1}}{\frac{y^3}{ba}} = \frac{ba}{y}$; hence, the w's and the y's are reciprocally proportional as was to be proved.

[27] This was obtained in the form omn.$p = y^2/2$, previous to October, 1674, from the Pascal form of the characteristic triangle; it is quoted as a known theorem in the essay dated 29 October, 1675. See §§ III, VI.

It is probably at this date that he began to revise his ideas as to d diminishing the dimensions; being forced to reconsider them by the occurrence of such equations as $wz = y$. It is seen in the next paragraph how careful he is to keep his dimensions equal; for he introduces an apparently irrelevant $a (= 1)$ for this purpose. It gradually dawns on him that neither $\int$ nor d alter the dimensions, but that a "sum of lines" is really a sum of rectangles, on account of the fact that they are applied in a certain fixed way to an axis; he is not quite certain of this however until well on in the next year, when we find him using $\int dx\, y$.

The artifice of this analysis[28] consisted in obtaining the abscissa from the ordinate; and this idea was never previously thought of. It is not a more difficult question either, if the curve is required in which BP, the interval between the normals and the ordinates, is reciprocally proportional to the abscissae AB. Indeed, $w = a^2/x$; but $w = y^2/2$; hence, we have

$$y = \sqrt{2\int w} \text{ or } \sqrt{2\int \frac{a^2}{x}}.$$

Now $\int w$ cannot be found except by the help of the logarithmic curve.[29] Hence, the figure that is required is that in which the ordinates are in the subduplicate ratio of the logarithms of the abscissae; and this curve is one of the transcendental curves.

Now, in truth, it is a much harder question,[30] if the curve, in which AP is reciprocally proportional to the ordinate BC is required.

For then $\quad x + w = \frac{a^2}{y}$ and $wz = \frac{y^2}{2d}$; also $\int z = x$,

or $\quad z = \frac{x}{d}$; thus, $w\frac{x}{d} = \frac{y^2}{2d}$, and $w = \frac{y^2}{2d} \cup \frac{x}{d}$;

hence, $\quad x + \frac{y^2}{2d} \cup \frac{x}{d} = \frac{a^2}{y}$.

If we suppose that the x's are in arithmetical progression then $x/d = z$ will be constant, and we shall have

$$x + \frac{y^2}{2d} = \frac{a^2}{y} \text{ or } \int x = \int \frac{a^2}{y} - \frac{y^2}{2},$$

therefore

$$\frac{x^2}{2} + \frac{y^2}{2} = \int \frac{a^2}{y} \text{ or } d\,\overline{x^2 + y^2} = \frac{2a^2}{y};$$

[28] It is difficult to see exactly what Leibniz means by this statement; I can only guess at substitution by means of the theorem $wz = y$, the equivalent to the recognition of the fact that $y\,dy/dx.dx = ydy$. The wording is however impersonal, and may mean that he himself had never thought of the idea before.

[29] Required $y = f(x)$, such that $y\,dy/dx = a^2/x$; the solution is $y^2 = 2a^2 \log_e Ax$. Weissenborn remarks on the omission of the a as being incorrect; from Leibniz's standpoint I cannot agree with him. Leibniz, from Mercator's work, connects a^2/x with the ordinate of the equilateral hyperbola $xy = a^2$, and its integral with the quadrature of this curve. The omission of the a^2 only alters the base of the logarithm, and Leibniz merely states that the solution is of a logarithmic nature without attempting to give it exactly.

[30] How does he know until he has tried it? This rather combats the idea that these were mere exercises; it gives this essay the appearance of being a fair copy intended either for publication or for one of his correspondents. If this were the case, the errors later in algebraical work are all the more unintelligible. The idea that Leibniz was a man who was accustomed to writing down his thoughts as he went along does not appeal to me at all; this is the method of the slow-working mind, rather than that of genius.

but, if we join AC, A(C), then these are equal to $\sqrt{x^2+y^2}$; and if with center A and radius AC we describe an arc CE to cut the straight line AE(C) in E, then E(C) will be the difference between AC and A(C); that is, $E(C)=e=d\overline{x^2+y^2}$

$$\therefore\ e=2a^2/y.$$

If then it were allowable to assume that the y's were also in arithmetical progression, we should have what was required; yet it seems that it does not make any difference even if the x's have been assumed to be in arithmetical progression. For if we do assume that the x's are in arithmetical progression, it follows that the AD's, or the y's are the reciprocals of the E(C)'s or the e's. Moreover, if they are so at any one time they are so at all times. Also, the sums of an infinite number of reciprocal proportionals, no matter what the progression may be of which they are taken as the reciprocal proportionals; for in this case there is not any consideration of rectangles, where there is need of equal altitudes, but a sum of lines is calculated, that of all the E(C)'s.[31] Hence I see the difficulty arise from the fact that the sum of every e, or every $2a^2/y$, or every E(C), cannot be obtained, unless we know to what progression the y's belong. In this case, that information is not given; for it is necessary that the x's should be in arithmetical progression, and hence that the y's are not so.

On the other hand, if we suppose in the above equation,

$$x+\frac{y^2}{2d} \smile \frac{x}{d}=\frac{a^2}{y},$$

that the y's are in arithmetical progression, then we have

$$x+\frac{y}{dx}=\frac{a^2}{y} \text{ or } xy+\frac{y^2}{dx}=a^2;$$

and, finally, by assigning the progression to neither x nor y, we have in general

$$xy+y\,\frac{d\frac{y^2}{2}}{dx}=a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

But we have not as yet really obtained anything. Let us therefore consider it from the standpoint of "indivisibles"; let PCS produced meet AD in S; then the sum of every AP applied to AB

[31] This seems to be the root of the error into which he falls; he has not yet perceived that the e's have to be *applied to some axis,* before he can sum them; and this is to a great extent due to the omission of the dx, taken as constant and equal to unity. He is thus bound to fall back on the algebraical summation of a series.

is equal to the sum of every AS applied to AD;[32] or calling DS, v, we have

$$dy\int y + dy\int v = dx\int x + dx\int w,$$
$$\text{or } dy\int y + dy\int v = dx\int a^2/y,$$

by the hypothesis of the question.

Now, if we take the y's to be in arithmetical progression, we have

$$\frac{y^2}{2} + \frac{x^2}{2} = dx \text{ Log} y.\ ^{33}$$

But just above, making the same supposition that the y's were in arithmetical progression, we had

$$xy + \frac{y^2}{dx} = a^2 \text{ or } dx = \frac{y^2}{a^2 - xy};$$

and now we have

$$dx = \frac{x^2 + y^2}{2 \text{ Log} y}.$$

Hence at length we obtain an equation, in which x and y alone remain, and unshackled, namely

$$\overline{y^2 + x^2},\ a^2 - yx = 2y^2 \overline{\text{Log } y};$$

and this equation, since it is determinate, will give the required locus.

This then is an exceedingly remarkable method, for the reason that when it is not in our power to have as many equations as there are unknowns, yet often we shall be able to obtain some more equations, by the help of which we shall be able to eliminate certain terms, as the term dx in this case, which alone stood in our way. Either of the two equations, by itself, contained the whole nature of the locus, although from neither of them could the solution be derived, because so far easy means were lacking; yet the combination of the two equations gave the solution at once.

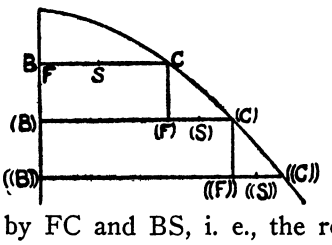
I see that the same thing could be otherwise obtained by moments; and here there comes to my mind a new consideration that is not altogether inelegant.

[32] From the characteristic triangle, AS : AP $= dx : dy$.

[33] This is of course nonsense. The error seems to arise from the dx being placed outside the integral sign; thus he assumes that dx is constant, while, for the integration, he also assumes that the dy is constant.

We cannot argue from this equation that Leibniz did not at this date appreciate what an infinitesimal was, on account of the infinitesimal being equated to a finite ratio; for since he is assuming that dy is an infinitely small unit, dx really stands for dx/dy.

In the attached figure, let $BC = y$, $FC = dy$; let S be the middle point of FC; then it is evident that the moment of FC is the



rectangle contained by FC and BS, i. e., the rectangle BFC; this follows from the fact that it is equal to BFC + SFC, and the latter can be neglected as being infinitely small compared to the former.[84]

Hence $\int y\,dy = y^2/2$, or the moment of all the differences FC will be equal to the moment of the last term, and $y\,dy = d(y^2/2)$, or $y^2 dy = y\,dy^2/2$.

Now, just above, in equation (A), by making x arithmetical, we had

$$y\,d\frac{y^2}{2} = a^2 - xy\text{, or } d\frac{y^2}{2} = \frac{a^2 - xy}{y};$$

but this is the same thing as $y\,dy$; hence $y\,dy = \frac{a^2 - xy}{y}$, and therefore $\int \overline{y\ \overline{dy}} = \int \overline{\frac{a^2}{y}} - \frac{x^2}{2}$. But we have already found that $\int \overline{y\ \overline{dy}} = \frac{y^2}{2}$; therefore $y^2 + x^2 = 2\int \overline{\frac{a^2}{y}}$, as before; i. e., $d\overline{x^2 + y^2} = \frac{2a^2}{y}$.

From this there follows something to be noted about these equations, in which occur $\int$ and d, where one quantity, in this case for instance the x, is taken to proceed arithmetically, namely, that we cannot make a change, nor say that the value of x is found, thus, $x = 2(a^2/y) - d\,\overline{y^2}$; for $d\,\overline{y^2}$ cannot be understood unless the nature of the progression of the y's is determinate. But the progression of the y's, in order that it may be used for $d\,\overline{y^2}$, must be such that the x's are in arithmetical progression; hence the dy's depend on the x's, and therefore the x's cannot be found from the dy's. For the rest, by this artifice many excellent theorems with regard to curves that are otherwise intractable will be capable of being investigated, namely, by combining several equations of the same kind.

In order that we may be better trained for really very difficult

[84] Note the advance in ideas suggested by the words "infinitely small *compared with* the former." Here, of course, the notation BFC is the usual notation of the period for BF.FC, the rectangle contained by BF and FC.

considerations of this kind, it will be a good thing to attempt just one more, as for instance when the AP's are reciprocally proportional to the AB's.

Here $x + w = \frac{a^2}{x}$, and $zw = d\frac{y^2}{2}$, and $z = dx$; and so we obtain

$$w = \frac{d\overline{\frac{y^2}{2}}}{z} = \frac{d\overline{\frac{y^2}{2}}}{dx}, \text{ hence } x + \frac{d\overline{\frac{y^2}{2}}}{dx} = \frac{a^2}{x}.$$

The solution of this is not now difficult; for if we suppose that the x's are arithmetical,[35] we have

$$\int x + \frac{y^2}{2} = \int \frac{a^2}{x}, \text{ or } x^2 + y^2 = \overline{\text{Log}\, y}.$$ [36]

Hence, $\sqrt{x^2+y^2} = \text{AC} = \sqrt{2\,\text{Log}\,\text{AD}}$; and this is a simple enough expression for the curve. In this however the AP's are required to be in arithmetical progression; but on the other hand, if the y's are taken to be in arithmetical progression, we have $x + y/dx = a^2/x$; and from this latter the nature of the curve is not easily obtained.

Let us see whether there can be a curve in which AC is always equal to BP; in this case $\sqrt{x^2+y^2} = w$, and $w = dy^2/2dx$. Let the x's be in arithmetical progression then $(\int \sqrt{x^2+y^2} =)$ $\int \text{AC} = y^2$; this, however, is not sufficient to describe the curve practically, that is to say, by points following one another consecutively. When $x = 1$, let $\text{BC} = (y)$; then $\sqrt{1 + (y^2)} = (y^2)$, or $1 + (y^2) = (y^4)$. Whence (y) may be obtained; thus, from the equation

$$y^4 - y^2 + \tfrac{1}{4} = 1 + \tfrac{1}{4}, \text{ we have } (y^2) = \frac{\sqrt{5}}{2}, \text{ or } (y) = \frac{\sqrt[4]{5}}{\sqrt{2}}.$$ [37]

Further, in the same way,

$$\underset{\text{AC}}{\sqrt{4 + ((y^2))}} + \underset{\text{A(C)}}{\sqrt{1 + \frac{\sqrt{5}}{2}}} = ((y^2));$$

and thus again $((y))$ can be found. By the help of this a third

[35] Note in general that this is Leibniz's equivalent of the modern phrase, "integrate with respect to x."

[36] This I think is more likely to be a slip on the part of Leibniz, than a misprint; for in the next line he has AD, which is the correct equivalent of y. Further, AP varies inversely as x, hence the AP's have to be in harmonical progression, not arithmetical, otherwise x is not equal to $x^2/2$. If on the other hand, we assume three errors of transcription, and replace x for y, AB for AD, AB for AP, the whole thing is correct with an arbitrary base.

[37] It is hardly necessary to point out the error in the arithmetical solution of the quadratic; nor is it important. It is however to be noted that if $\text{AC} = v$, the equation reduces to $v^2 = x(x + v)$, and the solution is a pair of straight lines.

AC can be found, and some sort of polygon can be found, which is more and more like the curve that is required, in proportion as the thing taken for unity is less and less.

That the x's are in arithmetical progression signifies that the motion (in describing it) along the axis AB is uniform. But descriptions that suppose any motion to be uniform are not within our power.[38] For we cannot produce any uniform motion, except a continually interrupted one.

Let us now examine whether $dx\,dy$ is the same thing as $d\overline{xy}$, and whether dx/dy is the same thing as $d\frac{x}{y}$; it may be seen that if $y=z^2+bz$, and $x=cz+d$; then

$$dy=z^2+2\beta z+\beta^2,\ +bz+b\beta,\ -z^2-bz,$$

and this becomes $dy=\overline{2z+b}\beta$.

In the same way $dx=+c\beta$, and hence

$$dx\,dy=\overline{2z+b}\;c\beta^2.$$

But you get the same thing if you work out $d\overline{xy}$ in a straightforward manner. For in each of the several factors there is a separate destruction, the one not influencing the other; and it is the same thing in the case of divisors.

Now let us see if there is any distinction when we seek the sums of these things. We have $\int dx=x$, $\int dy=y$, and $\int d\overline{xy}=xy$. If then we have an equation, $dx\,dy=x$ say, then $\int dx\,dy=\int x$. But $\int x=x^2/2$, hence $xy=x^2/2$, or $x/2=y$; and this satisfies the equation $dx\,dy=x$; for substituting for y its value, $ax\frac{dx}{2}=x$, or $a\frac{x^2}{2}=x$,[39] which is known to be true.

In sums these results do not hold good; for $\int x \int y$ is not the same thing as $\int xy$; the reason is that a difference is a single quantity, while a sum is the aggregation of many quantities. The sum of the differences is the latest term obtained. However, from the sums of the factors we can find the sums of products, not indeed as yet analytically, but by a certain method of reasoning; such as Wallis has done in this class of thing, not by proving them, but by a happy method of induction. Nevertheless to find proofs for them would be a matter of great importance.

[38] This is strongly reminiscent of Barrow, Lect. I (near the beginning) and Lect. III (near the end).

[39] Leibniz, as a logician, should have known better than to trust a single example as a verification of an affirmative rule.

With regard to infinitesimals note the equation $dx\,dy=x$!

Suppose $\int \overline{zy}$ to be the sum that is required. Let $\int \overline{zy} = w$, then $zy = \overline{dw}$, and $y = \dfrac{dw}{z}$, and $\int y = \int \overline{\dfrac{dw}{z}}$. Similarly, $\int z = \int \overline{\dfrac{dw}{y}}$. Suppose that $\int y$ is known, $= v$, and that $\int z$ is known, $= \psi$; then $y = dv = \dfrac{dw}{z}$, and $z = d\psi = \dfrac{dw}{y}$, and $\dfrac{dv}{d\psi} = \dfrac{z}{y}$. From this it would seem to follow that $d\dfrac{v}{\psi} = \dfrac{z}{y}$, and therefore that $\dfrac{v}{\psi} = \int \dfrac{z}{y}$. Therefore $\int \dfrac{z}{y} = \dfrac{\int z}{\int y}$, which is obviously incorrect.[40] Hence it follows that $\int \dfrac{dv}{d\psi}$ cannot be equal to $\dfrac{v}{\psi}$.

What then can it be? We have to sum the difference for v divided by the difference for ψ. That is, not every one of the differences for, or the whole of, v is to be divided by each single difference for the ψ; this is not so, I say, because each single one of the first set is only divided by the single one of the other set that corresponds to it, and not by all of them. Therefore $\int \dfrac{dv}{d\psi}$ is not the same as $\dfrac{\int dv}{\int d\psi}$, or $\dfrac{v}{\psi}$. Will not then $d\dfrac{v}{\psi}$ be something different from $\dfrac{dv}{d\psi}$? If it is the same, then also $\int d\dfrac{v}{\psi} = \int \dfrac{dv}{d\psi}$, that is $\dfrac{v}{\psi} = \int \dfrac{dv}{d\psi} = \dfrac{\int dv}{\int d\psi}$, which is absurd.

Similarly, if we can suppose that $\overline{dv\psi} = dv\ d\psi$, then $\int \overline{dv\psi}$, or $v\psi = \int \overline{dv\ d\psi}$. Now $v\psi = \int dv \int d\psi$; hence, $\int \overline{dv\ d\psi} = \int dv \int d\psi$; which is absurd.

Hence it appears that it is incorrect to say that $dv\,d\psi$ is the same thing as $dv\psi$, or that $\dfrac{dv}{d\psi} = d\dfrac{v}{\psi}$; although just above I stated that this was the case, and it appeared to be proved. This is a difficult point. But now I see how this is to be settled.

If we have v and ψ, and they form some quantity, say $\phi = v\psi$ or v/ψ, and if the values of v and ψ are expressed as rationals in terms of some one thing, for instance, in terms of the abscissa x, then the calculus will always show that the same difference is produced, and that $d\phi$ is the same as $dv\,d\psi$ or $dv/d\psi$. But now I see

[40] If Leibniz can see that this equality is "obviously incorrect," what is the use of the argument that has preceded this sentence; for the final result must also be obviously incorrect.

the former can never happen, nor can it come to the latter by separation of parts; for example,

$$x+\beta, \frown x+\beta,, -, x, x, \text{ becomes } 2\beta x,$$

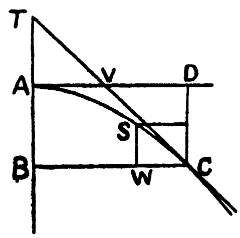
which is quite a different thing from

$$x+\beta, -x,, \frown x+\beta, -x \text{ which gives } \beta^2.$$

Hence it must be concluded that $dv\psi$ is not the same as $dv\,d\psi$, and

$$d\frac{v}{\psi} \text{ is not the same as } \frac{dv}{d\psi}. \quad [41]$$

Take an equation of the first degree, $a+bx+cy=0$. Let $\mathrm{DV}=\theta$, $\mathrm{AB}=x$, $\mathrm{BC}=y$, and $\mathrm{TB}=t$. Then, by making use of the method of tangents,[42] we have $bt=-cy$, or $t=-cy/b$. In the same way, $\theta=-bx/c$.



Let $\mathrm{WC}=w$, and $\mathrm{WS}=\beta$, then it is evident that $t/y=\beta/w$, and

$$\text{therefore } w=-\beta\frac{b}{c}, \text{ and in the same way, } \beta=\frac{-wc}{b}.$$

Second degree. $a+bx+cy+dx^2+ey^2+fyx=0$. Making use of the method of tangents, we have

$$bt+2dxt+fyt=-cy-2ey^2-fyx;$$

[41] Leibniz here justifiably verifies the falsity of his supposition being a general rule by a single breach of it. He uses $v=\psi=x$, and changes x into $x+\beta$; thus,

$$d(xx) = (x+\beta)(x+\beta)-xx \quad = 2\beta x$$
$$dx\ dx = (x+\beta-x)(x+\beta-x) = \beta^2.$$

Here we see the first idea of the method that is the same as that used by Fermat and, afterward by Newton and Barrow; this consideration, whatever the source, is that which leads him later to the substitution $x+dx$, $y+dy$ in those cases in which Barrow uses a and e.

[42] "ordinando et accommodando," literally setting in order and adapting. It is to be remembered that Sluse gave only a rule, and not a demonstration of the rule. Part of the rule was that, if the equation in two variables contained terms containing both the variables, these terms had to be set down *on each side of the equation.* Thus, for the equation $y^3=bvv-yvv$ would first of all be written

$$y^3+yvv=bvv-yvv \ldots\ldots\ldots\ldots \text{ ordinando (?)}$$

then each term on the left is multiplied by the exponent of y, and each term on the right by that of v, thus,

$$3y^3+yvv=2bvv-2yvv \ldots\ldots\ldots \text{ accommodando (?)}$$

and finally one y on the left, in each term, is changed into a t, where t is the subtangent measured along the y axis.

hence $t = \dfrac{-cy - 2ey^2 - fyx}{b + 2dx + fy}$. From this it is quite evident that t can always be divided by y (and θ by x), and since $w = \beta y/t$, therefore we have

$$w = \frac{\beta\overline{b + 2dx + fy}}{-c - 2ey - fx}, \text{ and } y = \frac{-w\,\overline{c + fx}, \frown \beta\overline{b + 2\ dx}}{f + 2e},$$

but from just above $y = \dfrac{-a - bx - dx^2}{c + ey + fx}$, hence we have

$$y = \frac{-w, \overline{c + fx}, \frown \beta\overline{b + 2\ dx}, \frown \overline{c + fx},,, + \overline{f + 2e} \frown \overline{a + bx + dx^2}}{-w, \overline{c + fx}, \frown -\beta\overline{b + 2dx}, \frown -e} \tag{43}$$

$$= \frac{-w\,\overline{c + fx}, -\beta\overline{b + 2dx}}{f + 2e}.$$

Hence we have an equation in which there is no longer any y;[44] and all figures that can be formed from this equation by a variation of the letters that stand for the constants can be squared; and also all others that by other methods can be shown to be connected with it.

§ IX.

In the manuscript that follows we must refrain from being critical; for, as suggested by the opening remark, it contains nothing more than random notes, jotted down as they came into Leibniz's mind, as materials for further investigation. In the ten days that have intervened since the date of the last MS., he has either had no spare time for further work on the lines of this last manuscript, or else he has found that he cannot proceed any further use-

[43] This is hopelessly inaccurate; all except one error, namely, $f + 2e$, which should be $\beta f + 2ew$, may be put down to bad transcription. Even if Leibniz's writing were execrable, the correct version of an ambiguous sign (through bad writing) could easily have been settled, *by working through the algebra*. Thus the first of the last pair of values, in Leibnizian symbols should be

$$y = \frac{-w, \overline{c + fx}, -\beta, \overline{b + 2dx},, \frown \overline{c + fx},,, + \overline{\beta f + 2ew}, \overline{a + bx + dx^2}}{-w, c + fx, -\beta, b + 2dx,, \smile -e,}$$

with a similar correction in the second value.

[44] Even if Leibniz had worked out the correct result, and obtained what he was trying for, namely, w/β in terms of x, he would have got a very lengthy quadratic, and the roots would be quite beyond his power to use at any time. But he convinces himself that he can thus find the quadrature of any conic, or figures that can be reduces to them.

fully until he has perfected the method he had in hand. He therefore reverts to the method of breaking up the figure into triangles by means of a set of lines meeting in a point, coupled with the ideas of the moment and the center of gravity, in order to try to obtain further general theorems for *analytical use*. In this way, he again comes across the differentiation of a product in the form of an "integration by parts"; but he does not recognize in it the differentiation of a product, for he says that as he has obtained this before he can get nothing new from it. He is still wasting his energies over the idea of obtaining dy/dx as an explicit function of x, for the purposes of *integration* or quadratures. The fact that he can use the method of Slusius *as an unproved rule* seems to have hidden from him the necessity of pushing on his investigations with regard to the laws of *differentiation*, or the direct tangent method.

21 November 1675.

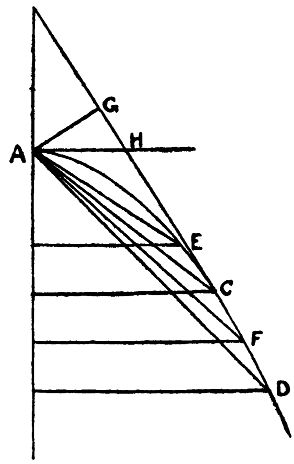
Pro methodo tangentium inversa et aliis tetragonisticis specimina et inventa. Trigonometria indivisibilium. Aequationes inadaequatae. ordinatae convergentes. Usus singularis Centri gravitatis.

[Examples and discoveries by means of the inverse method of tangents and other quadratures. Trigonometry of indivisibles. Inadequate equations. Converging ordinates. Special use of the Center of Gravity.]

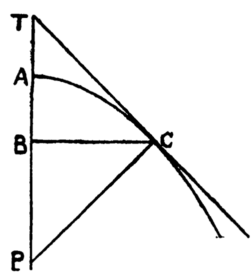
Subject-matter for a new consideration of the Center of Gravity method, as follows:

A segment AECD having been broken up into infinite triangles, AEC, ACF, etc., let the center of gravity of each of these triangles be found; this is a simple matter, for the center of gravity is always distant from the base a third of the altitude. Then, since the path of the center of gravity multiplied by the area of the triangle is equal to the solid formed by its rotation, and also since the products of the AH's and the infinitesimal parts of the axis are twice the areas of the triangle, also it is plain that the AG's multi-

plied by the distances of the centers of gravity of the triangles AEC from the axis are equal to the moment of the segment about the axis; by the help of this idea a number of things can be at once obtained in two ways: first, by taking some general figure and making a general calculation, and then so expressing it that the center of gravity can be easily found; in this way we may obtain the moments of spaces which would be a matter of difficulty otherwise, if they were investigated by the ordinary method of ordinates.



Secondly, on the other hand, if figures of which the moments are easily obtained in the ordinary way are treated by this method, we shall arrive at certain very difficult curves, the dimensions of which can always be deduced from some that are easier. Here then we have a remarkable rule, by the help of which useful properties can always



be obtained from any method however complicated. It is often useful when problems arise that we know are naturally simple, and from other reasons are soluble; for thus many notable cases are discovered. See what Tschirnhaus noted about the Hastarian line.

In irregular problems, such as cannot be treated in a straight-

forward manner or reduced to an equation that is sufficiently determinate, because, say, something has to be done inversely, it is useful to compare several ways with one another, of which the results should be identical. This seems to be useful for the inverse tangent method. Here is a case in point.

The figure, in which BP and AT are reciprocally proportional, is required.

Let $TB=t$, then $AT=t-x$, and $BP=a^2/(t-x)$. If this is multiplied by t, we have

$$\square TBP = ta^2/(t-x) = a^2 + a^2x/(t-x) = y^2$$

hence. $$ta^2 = ty^2 - xy^2,$$

or $t=xy^2/(a^2-y^2)$;[45] and therefore $t/x=y^2/(a^2-y^2)$, or all the t's together equal the moment about the vertex of every $y^2/(a^2-y^2)$.

But from other reasons, all the TP's applied to the axis are equal to the TC's applied to the curve.

Now $t/y=\beta/w$, and therefore $w=\dfrac{\beta y}{t=\dfrac{y^2x}{a^2-y^2}}=\dfrac{\beta\overline{a^2-y^2}}{xy}$

But $\int w=y$, therefore

$$\int\overline{\frac{\beta\overline{a^2-y^2}}{xy}}=y\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Further, $$wx=\frac{\beta\overline{a^2-y^2}}{y},\ \text{and}\ \int\overline{wx}=yx-\int y\beta,$$

hence, $$\int\frac{\beta\overline{a^2-y^2}}{y}=yx-\int y\beta\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

Also $$w=dy,\ dy=\frac{\beta\overline{a^2-y^2}}{xy},\ \text{and therefore}$$

$$xy=\frac{\beta\overline{a^2-y^2}}{dy=w}=\int y\beta+\int\frac{\beta\overline{a^2-y^2}}{y}.$$

Now if we suppose that the y's are in arithmetical progression, then $w=dy$ is constant and β is variable;

hence, $$\beta=\frac{\int\overline{y\beta+\beta\dfrac{a^2-y^2}{y}}}{a^2-y^2},\ d\,\overline{\beta\overline{a^2-y^2}}=\frac{a^2\beta}{y}.$$

But from equation (B), $\beta\dfrac{a^2-y^2}{y}+\beta y=\overline{dyx}$

hence, $$\beta\frac{a^2}{y}=\overline{dyx}.$$

[45] There is a mistake in sign; a^2-y^2 should be y^2-a^2; hence the work that follows is also wrong.

We have thus obtained two equations that are mutually independent, the first $\frac{dx}{dy} = \frac{yx}{a+y,\ a-y}$ [46] ....(1)

and the second $\overline{dyx} = \frac{dx\,a^2}{y}$. ....(2)

Let us seek to obtain others in addition, such as

$$\int t\,dy = \int y\,dx.$$

Now this furnishes us with nothing new; but $\int tw + \int xw = xy$ or $t\,dy + x\,dy = \overline{dxy}$, and $t = \frac{dx}{dy}\,y$; hence the latter $= \frac{\overline{dxy} - x\,dy}{dy}$. Therefore $\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}$.

Now this is a really noteworthy theorem and a general one for all curves. But nothing new can be deduced from it, because we had already obtained it.

However, from another principle we shall obtain a new theorem; for it is known that the sum of every $BP = BC^2/2$; that is to say, $BP = \frac{a^2}{t-x}$, $t = \frac{\beta y}{w} = \frac{\overline{dx}}{dy}\,y$, and therefore

$$BP = \frac{a^2\,dy}{\overline{dx}\,y - \overline{dy}\,x} = \frac{\overline{dy^2}}{2} \quad \text{................ (3)}$$

We therefore have two equations, in which dx occurs, namely, the first and the third; by the help of these, by eliminating dx, we shall have an equation in which only one of the unknowns remains shackled; thus from equation (1), we have $dx = \frac{\overline{dy}\,yx}{a^2 - y^2}$, and now from equation (3), we get $\overline{dx}\,y\,\overline{dy^2} - dy\,\overline{dy^2}x = 2a^2\,dy$. Hence,

$$dx = \frac{2a^2\,\overline{dy} + dy\,\overline{dy^2}x}{y\,\overline{dy^2}}.$$

We have therefore an equation between the two values of dx, in which only the y remains shackled. From this, by assuming

[46] Although the variables are separable, Leibniz does not recognize the fact that he can make use of this. For later he states that the solution of a problem cannot be obtained from a single equation. In this case we have

$$\frac{dx}{x} = \frac{y\ dy}{y^2 - a^2} = \frac{dv}{v}, \text{ if } y^2 - a^2 = \pm v^2.$$

Supposing this substitution to have been effected, Leibniz would have concluded that $x = v$, and would have stated that he had solved the problem.

But here again he has made an unfortunate choice, for the origin (A) cannot fall on any of the curves $Cx = v$ or $Cx^2 \pm y^2 = \pm a^2$, which is the general solution of the equation. Hence the problem is impossible.

the y's to be in arithmetical progression, that is that $dy=\beta$ a constant, and $\overline{dy^2}=z$, and $z=z^2/2=y^2$; $z=\sqrt{2}\,y=\overline{dy^2}$.[47] Thus we have obtained what was required.

We have here a most elegant example of the way in which problems on the inverse method of tangents are solved, or rather are reduced to quadratures. That is to say that the result is obtained by combining, if possible, several different equations, so as to leave one only of the unknowns in the tetragonistic shackle. This can be done by summing ordinates in various ways, or on the other hand, instead of ordinates, converging or other lines.

Note. If, instead of x or y, some other straight line can be found, either one that is oblique, or one of a number converging to the same point, by the employment of which one only of the unknowns is left in bonds, it may be employed with safety. Take for instance the case of finding the relation for the AP's; here the sum of AP's applied to the axis is half the square on AC. Whenever the formula for the one unknown that is left in shackles is such that the unknown is not contained in an irrational form or as a denominator,[48] the problems can always be solved completely; for it may be reduced to a quadrature, which we are able to work out; the same thing happens in the case of simple irrationals or denominators. But in complex cases, it may happen that we obtain a quadrature that we are unable to do. Yet, whatever it may come to, when we have reduced the problem to a quadrature, it is always possible to describe the curve by a geometrical motion; and this is perfectly within our power, and does not depend on the curve in question. Further, this method will exhibit the mutual dependence of quadratures upon one another, and will smooth the way to the method of solving quadratures. Meanwhile I confess that it may happen that there may be need for a very great number of inadequate equations (for so I call them, when there is need for many to solve the problem, although each alone would suffice provided it could be worked out by itself), in order to completely free one of the unknowns from its shackles. For, unfortunately, a solution cannot be obtained from a single equation, unless one of the terms is free from shackles; and if this term appears oftener, then not unless it is freed at least once. Thus there may be a great number

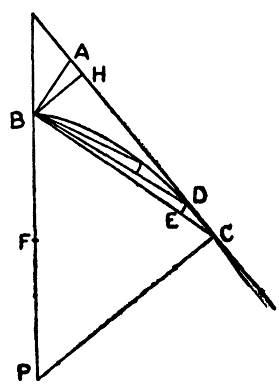
[47] This is quite unintelligible to me as it stands; query, is it an accurate transcription?

[48] This is tantamount to a confession by Leibniz that he cannot explicitly integrate $\int a^2/y$, although he knows that it is logarithmic or reduces to the area under the hyperbola; for he has given this in the MS. for Nov. 11.

of inadequate equations to be found; and we have to examine which of them are in some way independent of the others, i. e., such as cannot be derived from one another by a simple manipulation; for instance, the sum of all the AP's and the sum of all the AE's.

A new kind of Trigonometry of indivisibles, by the help of ordinates that are not parallel but converge.

Let B be a fixed point; let BDC be a very narrow triangle standing upon a curve; let DE be the perpendicular to BC; from the point B let BA, perpendicular to BC or parallel to DE, be drawn to meet the tangent AHDC, and let BH be the perpendicular to the tangent DC produced.



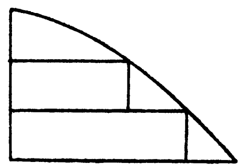
Then the triangles CED, CHB, BHA are similar; hence we have BH/CE = HA/DE = BA/CD, and therefore BH, DE = CE, HA, and BH, CD = CE, BH. Hence it follows that the sum of the triangles or the area of the figure is equal to the products of the AB's into the CE's, or the differences of the ED's and lastly AH,CD = DE, BH.[49]

Further, CH/CE = HB/DE = CB/CD; hence, again, CH, DE = CE, HB, and HB, CD = DE,CB; i. e., the area of the triangle, as is in itself evident, is equal to itself. Lastly, CH, CD = CE, CB; and this last result seems to be worth noting for the case of a Trochoid.

For, if by the rolling of a curve DC on a fixed plane CA, a trochoid curve is described by the point B fixed in DC, and it is given that the ordinate of the trochoid drawn to the fixed plane CA

[49] There are several errors in the letters in this paragraph, which are probably due to transcription; thus, an E for a (? badly written) B, an H for an A, etc., would be quite an easily-imagined error, provided the work was not verified during transcription.

is BH, then the sum of the intercepts CH applied to DC will be equal to the sum of the CB's applied to their own differences. Now if any ordinates are applied to their own differences, the same thing is always produced as in the case where we try to find the moment of the differences about the axis, which is the same as the moment when we take the sum of each, or the maximum ordinate, into the



distance of its center of gravity from the axis, i. e., its middle point, that is to say into half itself. Finally this is equal to half the square on the maximum ordinate. Therefore we can always obtain the sum of all the rectangles BC, CE, which is always equal to half the square on BC, or to the sum of all the BP's applied to the axis in F, where CP is the normal to the curve DC.

§ X.

Leibniz now directs his attention to the direct method of tangents, and proceeds to generalize the methods of Descartes. Is it only a coincidence that Barrow uses this method regularly, the curve that he is especially partial to being the rectangular hyperbola? Weissenborn suggests the same coincidence occurs with respect to the method of Newton, who uses analytical approximations; but if there is anything in either of these suggestions. I think that the Barrovian idea, which is purely for the construction of tangents, is much nearer to that of Leibniz in this manuscript than is the Newtonian.

However this may be, Leibniz is at last beginning to consider the point as to the method by which the principle of Sluse is obtained. He ascribes it to a development of the method of Descartes; but in this connection I cannot get out of my head the suggestion raised by Barrow's use of the first person *plural*, "frequently used by *us*," in the

midst of a passage that is written, contrary to his usual custom, in the first person *singular* throughout, where he describes the differential triangle and the "a and e" method. I consider that Sluse has enunciated a *working rule* for tangents, which he has generalized by observation of the results obtained by the use of the "a and e" method; and that this method had been circulated by Barrow some time before the publication of the *Lectiones Geometricae*, although I confess that I have not found any record of this, nor any distinct evidence of a correspondence between Barrow and Sluse; but there is more than a suggestion of this in the fact that Sluse's article was published in the *Phil. Trans.* for 1672.

It seems more than strange to me that there should be such a prolific crop of differential calculus methods within a couple of years of the work of Barrow in all sorts of places, raised by many different people, and that none of them allude to the general seed-merchant, as I consider Barrow to have been.

22 Nov. 1675.

Methodi tangentium directae compendium calculi, dum jam inventis aliarum curvarum tangentibus utimur. Quaedam et de inversa methodo.

[Compendium of the calculus of the direct method of tangents, together with its use for finding tangents to other curves. Also some observations on the inverse method.]

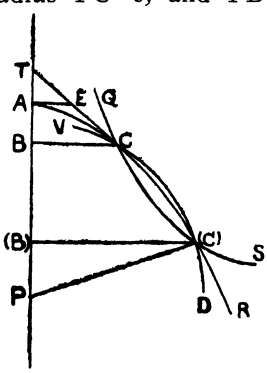
In that which I wrote on Nov. 21, I noted down those things which came to my mind concerning the method of tangents. Returning to the subject, let ACCR and QCCS be two curves that cut one another in one, two, or more points C, C; let AB(B) be the axis; let AB $= x$ be the ordinates, and BC $= y$ the abscissae; then we shall have two equations to the two lines, each in terms of these two principal unknowns. Now if these two equations have equal roots, or the equations have equal values, then the lines will touch one another. Instead of the line QCCS, Descartes chooses the arc of a circle VCCD, whose center is P, so that PC is the least of all

the lines that can be drawn from the point P. It will come to the same thing, and often more simply, if we take not the arc of a circle but the tangent line TC(C), that is the greatest of all those that can be drawn from a given point T to the curve. Let $TA=b$, $AE=e$, be assumed as given; required to find AB, BC. The two equations are, the one for the curve AC(C), namely, $ax^2+cy^2+\text{etc.}=0$, and the other to the straight line TC(C) which, on account of the relation $TA/AE=TB/BC$, will be $b/e=(b\pm x)/y$ or $\pm x=(b/e)y-b$

$$\text{or } y=\pm\,(e/b)x+e.$$

Thus the value of either one or other of the unknowns can always be obtained explicitly, and thus can be worked out immediately without raising the degree of the equation of the given curve AC(C); and then at once we shall obtain an equation giving the unknown that alone remains, so that we may determine the condition for equal roots. Doubtless this is the principle of Sluse's method.

If however the arc of the circle whose center is P is used, following Descartes, then the new equation, for the circle, will be as follows: let the radius $PC=s$, and $PB=v-x$, and we have



$s^2=y^2+v^2+x^2-2vx$. Hence it is clear that we have the choice of either a circle or a straight line; and when, in the equation to the given curve, only an even power of y appears (as can always be made to happen in the case of the conics), then it will be more convenient to use equations to circles; for thus, by the help of the two values of y^2, the unknown x can be immediately worked out; but, in general for all equations to curves expressed by a rational relation, the method of the straight line may be usefully employed.

Hence I go on to say that not only can a straight line or a circle, but any curve you please, chosen at random, be taken, so long as the method for drawing tangents to the assumed curve is

known; for thus, by the help of it, the equations for the tangents to the given curve can be found. The employment of this method will yield elegant geometrical results that are remarkable for the manner in which long calculation is either avoided or shortened, and also the demonstrations and constructions. For in this way we proceed from easy curves to more difficult cases, and an equation to a curve being supposed known, it is always possible to choose an equation to some other curve whose tangents are known, by the help of which one of the unknowns can be worked out very easily.

Thus, if it is given that $hy^2+y^3=cx^3+dx^2+ex+f$ is the equation to a curve of which the tangents are required, assume a curve of which the equation is $hy^2+y^3=gx+q$, for that of which the tangents are already known; eliminating y, we have an equation such as $gx+q=cx^3+dx^2+ex+f$. This can be determined for two equal roots, either by Descartes's method of comparisons, or Hudde's by means of an arithmetical progression; and thus by working out the value of x, the value of either g or q may be found; and one of the two letters q or g can be chosen arbitrarily.[50] Hence, a way of describing that other curve that touches the given curve is obtained; now, when this is described, let the tangent be drawn at the point which is common to it and the proposed curve, which tangents we have supposed to be already known; then this tangent will touch the given curve.

I think that, in general, the calculation will be possible by this method of assuming a second curve, as we have done in this case, which evidently works out one of the unknowns. Hence I fully believe that we shall derive an elegant calculus for a new rule of tangents, which in addition may be better than that of Sluse, in that it evidently works out immediately one of the two unknowns, a thing that the method of Sluse did not do. Now this very general and extensive power of assuming any curve at will makes it possible, I am almost sure, to reduce any problem to the inverse method of tangents or to quadratures. Indeed let any property of the tangents to a curve be given, and let the relation between the ordinates

[50] The method of Hudde appears to be similar in principle to that of Sluse, while that of Descartes was the construction of the derived function by assuming roots, forming the sum of the quotients of the function divided by each of the assumed root-factors in turn, and comparison with the original function. Both therefore reduce to finding the common measure of the equation to the curve (where the right-hand side is zero) and the differential of it.

Leibniz, however, strange to say, does not note that by taking one of his arbitrary constants, q, equal to f, the equation has its degree lowered in the particular case he has chosen.

and the abscissae be required. Then an equation can be derived, which will contain the principal unknowns, x, y, and always two others as incidentals, such as s and v, or b and e, or the like; now, as the equation contains the property of the tangents, by which s and b may be expressed so as to have a relation to the tangents, assume in this case any new curve chosen arbitrarily, and then s and v will also have some known relation to this curve. By means of the equation to the arbitrarily chosen curve, we shall be able to replace the given property of tangents in favor of the curve required, namely, by removing one or other of the unknowns; and by thus reducing the problem to such a state the inverse calculation will come out the more easily.

The whole thing, then, comes to this; that, being given the property of the tangents of any figure, we examine the relations which these tangents have to some other figure that is assumed as given, and thus the ordinates or the tangents to it are known. The method will also serve for quadratures of figures, deducing them one from another; but there is need of an example to make things of this sort more evident; for indeed it is a matter of most subtle intricacy.

The manuscripts mentioned above seem to be all that were found by Gerhardt belonging to the period 1673-5. I feel that it is a great pity that they were not given in full, or at least a little more fully. For instance, Gerhardt mentions that Leibniz in the MS. of August 1673 constructs the so-called characteristic triangle, but does not give Leibniz's figure in connection. This figure should have been given; for the figure given in October 1674 is not the characteristic triangle as given by Leibniz in the "postscript" (§ I), or the *Historia* (§II), but it *is the Pascal diagram* (assuming that the figure given by Cantor is the correct one). It would be useful to know the date at which Leibniz drops the Pascal diagram in favor of one or other of the Barrow diagrams.

It is to be noticed at this date that Leibniz uses one infinitesimal only, and *verifies* that the method of Descartes comes out correctly in the simple case of the parab-

ola; but he is not satisfied with the generality of the method of neglecting the vanishing quantities.

Again, the second manuscript of October 1674 appears to be immensely important; especially as it contains the groundwork of some of the later manuscripts. Judging by the little that is given of it, it would seem to be most desirable that fuller extracts, at least, should have been given. It is a matter for remark that this manuscript is a long essay on series. Can this possibly have had anything to do with the fact that it is not given in full?

(TO BE CONTINUED.)

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